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United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Srinivasa Ramanujan

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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Version 1.1, Aug 2019

1. Set the scene for your work on these sheets by discovering some facts about the life and work of the twentieth century mathematician Srinivasa Ramanujan.

SOLUTION

See the reference given on the front of both the Questions document and the Solutions document.

2. An extremely absent-minded logician meets two (unmarried) identical twin sisters called Tilly and Lilly. Tilly always tells the truth and Lilly always lies. The logician and one of the sisters fall in love and they marry. After the honeymoon, the logician has to go abroad for a couple of weeks and the bride's sister comes to stay with her to cheer her up during the separation.

a) On his return, the logician collected his wife from a party at the house of a friend where the sisters had spent the weekend. When he walked in he spotted one of them immediately but, of course, he had no idea whether or not she was his wife. (Suspend your disbelief: this is a logic problem and therefore possibly not a real-life situation.) The logician liked to make things difficult for himself so he resolved to identify his wife by asking the woman a three-word question requiring a yes / no answer. What question could he ask? Justify your choice.

b) The next day the logician went to work and, on his way home, he said to himself "I must find out, once and for all, my wife's first name". Since he had forgotten his key, he rang the bell and the door was opened by one of the sisters. What three-word question, requiring a yes / no answer can he ask this time? Justify your choice

c) Could the logician ask a single yes / no question (of any length) which would simultaneously resolve the identity of his wife and tell him what her first name is?

SOLUTION

a) The logician cannot ask the woman a direct personal question because he does not know whether she is the truth-teller or the liar. A more useful approach might be to ask a question about the existing situation. Suppose he asks "Is Tilly married?" Further suppose that he is indeed married to Tilly. If the woman addressed is Tilly, then she replies "Yes", because she is a truth-teller. If he is speaking to Lilly, then she replies "No", because she is lying about the fact that Tilly and the logician are married. What happens if the logician is actually married to Lilly? If the woman addressed is Tilly, then she truthfully replies "No". However, if he is speaking to Lilly, then she will lie and say "Yes". Note that, in both scenarios, the logician receives a "Yes" answer from the woman to whom he is actually married and this enables him to identify his wife.

b) This time a possible question is "Are you married?" Suppose the woman who opens the door replies "Yes". She is either Tilly or Lilly; suppose first that she is Tilly. Then her "Yes" is a a truthful reply. On the other hand, if the woman is Lilly, she falsely answers "Yes", although she is not married. In either case the "Yes" answer indicates that the name of the logician's wife is Tilly. Now check for yourself that a "No" answer implies that the name of his wife is Lilly.

c) No such question exists because there are four possibilities in this case. Either:
the woman addressed is his wife Tilly;

the woman addressed is his wife Lilly;
the woman addressed is his sister-in-law Tilly;
the woman addressed is his sister-in-law Lilly.
A yes / no question can elicit just two possible responses. This worked in the two previous situations but will not do so here.

3. Let ABC be a triangle and O be the centre of the circle passing through A, B and C (which has radius R). Define the vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$.

a) Write down expressions for $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b}$ in terms of R and the angles of the triangle.

b) Let H be the point such that $\overrightarrow{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Show that H lies on all three altitudes of the triangle (H is the *orthocentre* of the triangle). *Note: an altitude of a triangle is a line passing through a vertex which is perpendicular to the opposite side.*

c) Show that the three medians pass through a point, which we will denote by G (the *centroid* of the triangle). Further show that O, G and H are collinear and determine $\frac{OH}{OG}$. *Note: a median of a triangle is a line passing through a vertex and the midpoint of the opposite side.*

d) Show that there is a point N lying on the line OH, which is the centre of a circle passing through all three midpoints of the sides as well as the midpoints of segments AH, BH and CH.

e) Further show that circle in part (d) passes through the feet of the three altitudes. This circle is called the *nine-point circle*.

SOLUTION

Note: If you are working through any part of this solution, then it is strongly recommended that you draw plenty of diagrams to illuminate the arguments.

(a) For vectors $\mathbf{u}$ and $\mathbf{v}$ with angle θ between them, their *dot product* is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta,$$

and in particular $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$. Therefore

$$\mathbf{a} \cdot \mathbf{a} = |\overrightarrow{OA}|^2 = R^2,$$

and similarly $\mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c} = R^2$.

Now, using the definition, $\mathbf{a} \cdot \mathbf{b} = |\overrightarrow{OA}||\overrightarrow{OB}| \cos \angle AOB = R^2 \cos \angle AOB$. O is the centre of the circle passing through A, B and C so, as the angle at the centre of a circle is twice the angle at the circumference, $\angle AOB = 2\angle ACB = 2C$. Therefore

$$\mathbf{a} \cdot \mathbf{b} = R^2 \cos(2C),$$

and similarly $\mathbf{b} \cdot \mathbf{c} = R^2 \cos(2A)$, $\mathbf{c} \cdot \mathbf{a} = R^2 \cos(2B)$. (Note that here the angles of triangle ABC are abbreviated to A, B and C.)

(b) H is defined by $\overrightarrow{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and we want to show that it lies on all three altitudes. To show it lies on the altitude from A to BC we need to show that AH is perpendicular to BC. This can be done with the dot product: two vectors are perpendicular if and only if their dot product is zero. Now

$$\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = \mathbf{a} + \mathbf{b} + \mathbf{c} - \mathbf{a} = \mathbf{b} + \mathbf{c},$$

and

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{c} - \mathbf{b}.$$

Therefore

$$\begin{aligned}\overrightarrow{AH} \cdot \overrightarrow{BC} &= (\mathbf{c} + \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) \\ &= \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{c} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b} = R^2 - R^2 \\ &= 0,\end{aligned}$$

and so $AH \perp BC$ and hence H lies on the altitude from A to BC. The definition of H is symmetric in A, B and C so H lies on all three altitudes (alternatively we could check that $\overrightarrow{BH} \cdot \overrightarrow{CA} = \overrightarrow{CH} \cdot \overrightarrow{AB} = 0$). We have shown, using vectors, the existence of the orthocentre of a triangle.

Note: If you have worked on G H Hardy sheets you may have come across two other methods of proving that the altitudes of a triangle all pass through a single point.

(c) In order to work with medians we need to find the vectors of the midpoints of the sides: let A_1 be the midpoint of BC, B_1 the midpoint of CA and C_1 the midpoint of AB. Then we have

$$\overrightarrow{OA_1} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c}), \quad \overrightarrow{OB_1} = \tfrac{1}{2}(\mathbf{c} + \mathbf{a}), \quad \overrightarrow{OC_1} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

The median from A to BC is the line AA_1 so this has vector equation $\mathbf{r}_1 = \overrightarrow{OA} + t\overrightarrow{AA_1} = \mathbf{a} + t(\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \mathbf{a}) = (1-t)\mathbf{a} + \frac{t}{2}\mathbf{b} + \frac{t}{2}\mathbf{c}$ where t is a real number. Similarly the median BB_1 has vector equation $\mathbf{r}_2 = (1-s)\mathbf{b} + \frac{s}{2}\mathbf{a} + \frac{s}{2}\mathbf{c}$ where s is a real number.

We want to find the intersection of AA_1 and BB_1 and then show that this lies on the line CC_1. At the point where AA_1 and BB_1 meet s and t satisfy

$$(1-t)\mathbf{a} + \tfrac{t}{2}\mathbf{b} + \tfrac{t}{2}\mathbf{c} = (1-s)\mathbf{b} + \tfrac{s}{2}\mathbf{a} + \tfrac{s}{2}\mathbf{c}.$$

We could try and match up the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and hope that there is a pair of reals s and t for which these are all equal (a priori this might not work as we get three equations with two variables so there might not be a solution at all). Doing this we want $1 - t = \frac{s}{2}$, $\frac{t}{2} = 1 - s$ and $\frac{t}{2} = \frac{s}{2}$ from which we can deduce that $s = t = \frac{2}{3}$. Therefore the point of intersection, G, has position vector

$$\overrightarrow{OG} = \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

We check that G lies on the median CC_1 (you could skip this, arguing that the definition of G is symmetrical in A, B and C and that G lies on the median AA_1 so it must lie on all three medians). Note that $\overrightarrow{CG} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c}$ and $\overrightarrow{GC_1} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) - \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{6}\mathbf{a} + \frac{1}{6}\mathbf{b} - \frac{1}{3}\mathbf{c}$. Therefore $2\overrightarrow{GC_1} = \overrightarrow{CG}$ so that C, G and C_1 all lie on a line with $CG = 2GC_1$ (G divides the median CC_1 in the ratio $1 : 2$). Note also, by symmetry that G must divide each median in the ratio $1 : 2$.

We will show that O, G and H are collinear (this line is called the *Euler line*): we have $3\overrightarrow{OG} = \mathbf{a} + \mathbf{b} + \mathbf{c} = \overrightarrow{OH}$ so that O, G and H lie on a line with $\frac{OH}{OG} = 3$.

(d) We are told the point N lies on the line OH so we know that

$$\overrightarrow{ON} = t\overrightarrow{OH} = t(\mathbf{a}+\mathbf{b}+\mathbf{c}),$$

where t is some real number.

Let U, V, W be the midpoints of segments AH, BH, CH respectively so that

$$\overrightarrow{OU} = \mathbf{a} + \tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{c}, \quad \overrightarrow{OV} = \mathbf{b} + \tfrac{1}{2}\mathbf{c} + \tfrac{1}{2}\mathbf{b}, \quad \overrightarrow{OW} = \mathbf{c} + \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}.$$

We want to pick t so that N is equidistant from A_1, B_1, C_1, U, V and W. It is possible to guess that $t = \frac{1}{2}$ works or to guess that N is the midpoint of A_1U, B_1V and C_1W and then to check that this works. We will calculate t directly.

We have $\overrightarrow{NA_1} = (\frac{1}{2}-t)\mathbf{b} + (\frac{1}{2}-t)\mathbf{c} - t\mathbf{a}$ and $\overrightarrow{NB_1} = (\frac{1}{2}-t)\mathbf{c} + (\frac{1}{2}-t)\mathbf{a} - t\mathbf{b}$ therefore

$$\begin{aligned}|\overrightarrow{NA_1}|^2 &= (\tfrac{1}{2}-t)^2\mathbf{b}\cdot\mathbf{b} + (\tfrac{1}{2}-t)^2\mathbf{c}\cdot\mathbf{c} + t^2\mathbf{a}\cdot\mathbf{a} \\ &\quad + (\tfrac{1}{2}-t)^2\mathbf{b}\cdot\mathbf{c} - t(\tfrac{1}{2}-t)\mathbf{a}\cdot\mathbf{b} - t(\tfrac{1}{2}-t)\mathbf{c}\cdot\mathbf{a} \\ &= (\tfrac{1}{2}-t)^2R^2 + (\tfrac{1}{2}-t)^2R^2 + t^2R^2 \\ &\quad + (\tfrac{1}{2}-t)^2\mathbf{b}\cdot\mathbf{c} - t(\tfrac{1}{2}-t)\mathbf{c}\cdot\mathbf{a} - t(\tfrac{1}{2}-t)\mathbf{a}\cdot\mathbf{b}.\end{aligned}$$

Similarly

$$\begin{aligned}|\overrightarrow{NB_1}|^2 &= (\tfrac{1}{2}-t)^2R^2 + (\tfrac{1}{2}-t)^2R^2 + t^2R^2 \\ &\quad + (\tfrac{1}{2}-t)^2\mathbf{c}\cdot\mathbf{a} - t(\tfrac{1}{2}-t)\mathbf{a}\cdot\mathbf{b} - t(\tfrac{1}{2}-t)\mathbf{b}\cdot\mathbf{c}.\end{aligned}$$

We want these two quantities to be equal (as N is equidistant from A_1 and B_1) setting them equal and cancelling gives that we need

$$\tfrac{1}{2}(\tfrac{1}{2}-t)\mathbf{b}\cdot\mathbf{c} = \tfrac{1}{2}(\tfrac{1}{2}-t)\mathbf{c}\cdot\mathbf{a}$$

so we need

$$(\tfrac{1}{2}-t)\cos(2A) = (\tfrac{1}{2}-t)\cos(2B),$$

using our expressions from part (a). In a scalene triangle $\cos(2A)$ and $\cos(2B)$ will be distinct and so $t = \frac{1}{2}$.

Setting N as the point such that $\overrightarrow{ON} = \frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ we get that

$$\begin{aligned}&\overrightarrow{NA_1} = -\tfrac{1}{2}\mathbf{a}, && \overrightarrow{NB_1} = -\tfrac{1}{2}\mathbf{b}, && \overrightarrow{NC_1} = -\tfrac{1}{2}\mathbf{c}, \\ &\overrightarrow{NU} = \tfrac{1}{2}\mathbf{a}, && \overrightarrow{NV} = \tfrac{1}{2}\mathbf{b}, && \overrightarrow{NW} = \tfrac{1}{2}\mathbf{c},\end{aligned}$$

so the circle centre N radius $\frac{R}{2}$ passes through A_1, B_1, C_1, U, V and W. Even further we notice that N is the midpoint of UA_1, VB_1, WC_1 (since $\overrightarrow{NA_1} = -\overrightarrow{NU}$ etc).

(e) Let the feet of the altitudes from A, B and C be D, E and F respectively. It is tricky to calculate the position vectors of these points and so check that they lie on the circle from (d) by showing that $ND = NE = NF = \frac{R}{2}$. However we can use our observation that N is the midpoint of UA_1: this means that the circle from (d) has diameter UA_1. Therefore to show D lies on this circle we only need to show that $\angle UDA_1 = 90°$.

But D and A_1 are on BC is and U, D lie on the the altitude from A to BC so $UD \perp DA_1$. Thus D lies on the circle from part (d) and E and F do similarly.

4. You may have met Ms X MP and her intern Jane in G H Hardy, Sheet 8, Question 6; now Ms X has a new project for her assistant. "I had a very good socio-economic model to develop employment opportunities in my constituency. Unfortunately your predecessor did not collect all the relevant data and time was wasted." (Jane prudently keeps quiet.) Ms X goes on to explain that the model requires the solution of the following system of equations.

$$a + b = 18 \tag{1}$$
$$au + bv = 110 \tag{2}$$
$$au^2 + bv^2 = 690 \tag{3}$$
$$au^3 + bv^3 = 4430 \tag{4}$$

She adds that, when she was at school, she met only systems of linear simultaneous equations. Show how Jane can again reassure Ms X by solving the equations.

Solution

Jane can actually show Ms X that the sort of techniques that she already knows can be applied to this system of equations. First, multiplying (1) through by u and subtracting (2) yields

$$b(u - v) = 18u - 110 \tag{5}$$

Similarly, multiplying (2) through by u and subtracting (3) gives

$$bv(u - v) = 110u - 690 \tag{6}$$

And again, multiplying (3) through by u and subtracting (4) produces

$$bv^2(u - v) = 690u - 4430 \tag{7}$$

Now observe that the product of the left-hand sides of (5) and (7) is equal to the square of the left-hand-side of (6): they are both equal to $b^2v^2(u - v)^2$. Hence the same relation must hold between the right-hand-sides of these equations. Thus

$$(18u - 110)(690u - 4430) - (110u - 690)^2 = 0.$$

On multiplying out and rearranging (all without employing a calculator, of course) this reduces to

$$320u^2 - 3840u + 11200 = 0.$$

On dividing through by 320, we have

$$u^2 - 12u + 35 = 0.$$

This quadratic equation has roots $u = 7$ and $u = 5$. If u takes the value 7 then $v = 5$: this can be obtained as follows. Setting $u = 7$ on the right-hand of (5) gives $18u - 110 = 16$ and it follows that $b(u - v) \neq 0$. We can divide (6) by (5) to obtain $v = \dfrac{80}{16} = 5$. Then (5) yields $b = 8$ and substituting this value in (1) gives $a = 10$. It is left to the reader to obtain the corresponding values for the case $u = 7$.

Of course the validity of any of these figures depends on the quality of Ms X's model and the reliability of the data previously collected. We might be suspicious of the quadratic equation with its two values for u. We should also check that the values found do actually satisfy the equations. This is particularly recommended as the working involved squaring.

5. On the same axes, sketch the graphs

$$y = (x+1)^3 - (x+1) \qquad \text{and} \qquad y^2 = (x+1)^3 - (x+1).$$

You should justify all the features of the graphs that you show and give the co-ordinates of any significant points. In particular, give details of the points where the two graphs intersect.

SOLUTION

The graph $y = (x+1)^3 - (x+1)$ is a standard cubic, crossing the $x-$axis at $x = -2$, $x = -1$ and $x = 0$ and such that $y \to \infty$ as $x \to \infty$.
The graph has two turning-points, one a local minimum and one a local maximum. They occur where

$$\frac{dy}{dx} = 3(x+1)^2 - 1 = 0.$$

Thus the values of x at these turning-points are $x = -1 \pm \dfrac{1}{\sqrt{3}}$. The local minimum occurs when $x = -1 + \dfrac{1}{\sqrt{3}}$ and the local maximum occurs when $x = -1 - \dfrac{1}{\sqrt{3}}$. You can check these two statements in detail if you like. However, since we are after a sketch only, it suffices to deduce them from the general shape of the curve, as initially described.
When $x = -1 + \dfrac{1}{3\sqrt{3}}$, then

$$y = \frac{1}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}. \tag{1}$$

When$x = -1 - \dfrac{1}{\sqrt{3}}$, then

$$y = -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}. \tag{2}$$

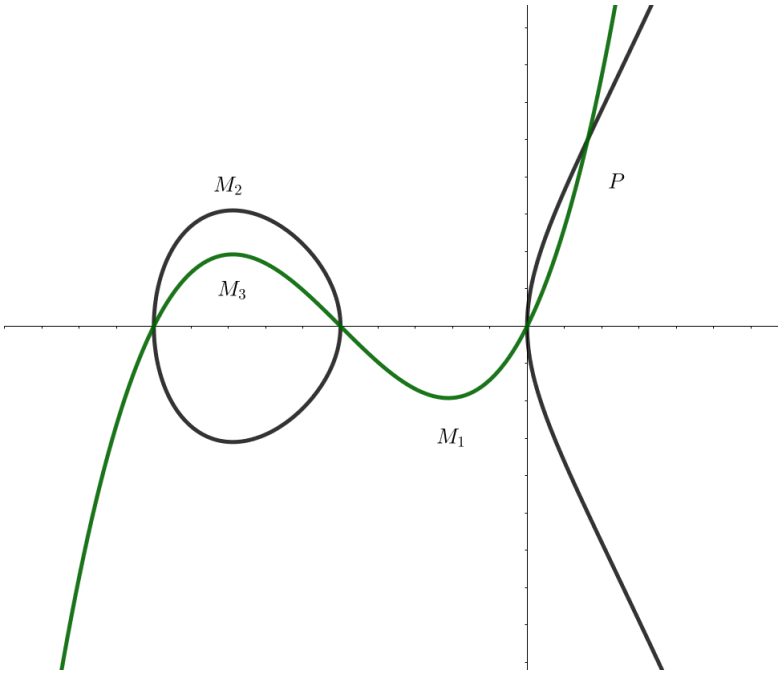
Turning to the other curve, we see that it combines the two graphs

$$y = \pm\sqrt{(x+1)^3 - (x+1)}.$$

Now each of these functions is defined only when $(x+1)^3 - (x+1) \geqslant 0$ and this inequality is satisfied for $x \geqslant 0$ and $-2 \leqslant x \leqslant -1$. On these intervals the graph $y = \sqrt{(x+1)^3 - (x+1)}$ has ***very*** roughly the same shape as $y = (x+1)^3 - (x+1)$. Once we have this graph, then we know that the graph $y = -\sqrt{(x+1)^3 - (x+1)}$ can be obtained by reflection in the $x-$axis. However, we are going to put these graphs onto the same axes as $y = (x+1)^3 - (x+1)$ and so we need to know the relative positions of the two curves. If $u \geqslant 0$, then $\sqrt{u} \geqslant u$ when $u \leqslant 1$. Now (2) shows that the $y-$value for the first graph at the local maximum is $\dfrac{2}{3\sqrt{3}} < 1$. (Make sure you understand why this is so.) Hence, when $-2 \leqslant x \leqslant -1$, all the $y-$values for the first curve are less than 1. Consequently the $y-$values for the (positive) square root curve lie above those for the first curve on this interval.

It remains to consider what happens when $x \geqslant 0$. Again, the square root curve will lie above the first curve when $0 \leqslant y \leqslant 1$. Thus the two curves will intersect when $y = 1$. The diagram below

shows some of the features discovered so far. The green curve is the first graph and the black curve shows the whole of the second graph, with the reflected portion added.



M_1 is the local minimum point on the first curve, so $M_1 = \left(-1+\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right)$.

Again M_3 is the local maximum point on the first curve, so $M_3 = \left(-1-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$.

M_2 is the local maximum point on the upper part of the closed portion of the black curve $y^2 = (x+1)^3 - (x+1)$. Differentiating implicitly gives

$$2y\frac{dy}{dx} = 3(x+1)^2 - 1. \tag{3}$$

This shows that both the turning-points on the black curve have the same $x-$coordinate as M_3, namely $\left(-1-\frac{1}{\sqrt{3}}\right)$. An extra input of calculation would produce the $y-$coordinates.

The implicit differentiation also illuminates the behaviour of the black graph near the origin. If $y \neq 0$, then (3) may be rearranged as

$$\frac{dy}{dx} = \frac{3(x+1)^2 - 1}{2y}. \tag{4}$$

As $x \to 0$, then $3(x+1)^2 - 1 \to 2$. Also $y^2 \to 0$ and hence $y \to 0$. Then (4) implies that $\frac{dy}{dx} \to \infty$.

One last detail that might be considered is the intersection point P. It has already been remarked that P has $y-$coordinate 1. Some judicious guessing, plus slightly laborious calculation shows that the $x-$coordinate lies between 0.3 and 0.4. Of course you could use a numerical method to improve this result. However, here is a short cut: if $x = \frac{1}{3}$, then, using the green curve, we have

$$y = \left(\frac{1}{3}+1\right)^3 - \left(\frac{1}{3}+1\right) = \frac{1}{27} + 3\times\frac{1}{9} + 2\times\frac{1}{3} = \frac{1}{27} + 1.$$

In other words, when $x = \frac{1}{3}$, then y differs from 1 by as little as $\frac{1}{27}$. Hence the $x-$coordinate of P is likely to be close to $\frac{1}{3}$.

6. The function f has the interval $[0, 1]$ (the real numbers between 0 and 1 inclusive) as its domain and range. It also satisfies $f(0) = 0$, $f(1) = 1$ and

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}. \qquad (*)$$

Find all possible values of $f\left(\frac{1}{2}\right)$ and $f\left(\frac{1}{3}\right)$.

SOLUTION

We are told the value of $f(x)$ at two points ($x = 0$ and $x = 1$) and wish to find its possible values at some other points. This is where (*) comes in handy: if we substitute in values of x and y for which we know $f(x)$ and $f(y)$, then we can deduce the value of $f\left(\frac{x+y}{2}\right)$.

Setting $x = 0$ and $y = 1$ in (*) gives

$$f\left(\frac{1}{2}\right) = \frac{f(0)+f(1)}{2} = \frac{1}{2}.$$

Now we could substitute $x = \frac{1}{2}$ and $y = 0$ in (*) to find the value of $f\left(\frac{1}{4}\right)$ and $x = \frac{1}{2}$ and $y = 1$ to find $f\left(\frac{3}{4}\right)$. We could continue in this way, using only those x and y for which we have already found the values of $f(x)$ and $f(y)$. However, this will not immediately give us $f\left(\frac{1}{3}\right)$.

Thus we might try something different. It might be helpful to take $x = \frac{1}{3}$ and play around with various equations. If we take $x = \frac{1}{3}$ and $y = 1$, then we get

$$f\left(\frac{2}{3}\right) = \frac{f\left(\frac{1}{3}\right)+f(1)}{2} = \frac{f\left(\frac{1}{3}\right)+1}{2}.$$

so we know $f(\frac{2}{3})$ in terms of $f(\frac{1}{3})$. If we now take $x = \frac{1}{3}$ and $y = \frac{2}{3}$, then $\frac{x+y}{2} = \frac{1}{2}$, which is excellent as we know what $f\left(\frac{1}{2}\right)$ is. So taking $x = \frac{1}{3}$ and $y = \frac{2}{3}$, we get

$$\frac{1}{2} = f\left(\frac{1}{2}\right) = \frac{f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)}{2}.$$

Taking the first and last terms and substituting for $f\left(\frac{2}{3}\right)$, we have

$$1 = f\left(\frac{1}{3}\right) + \frac{f\left(\frac{1}{3}\right)+1}{2}.$$

Solving this equation yields

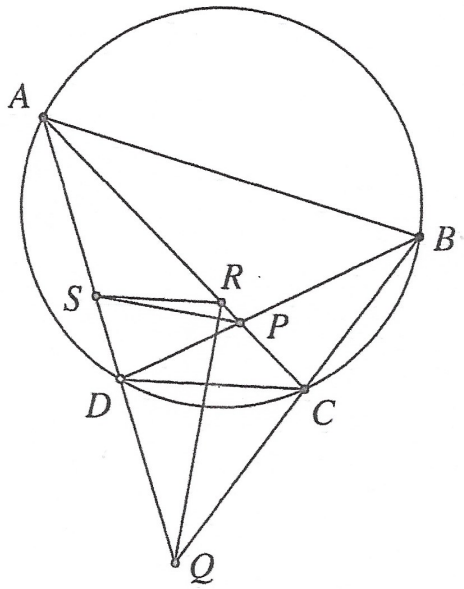
$$f\left(\frac{1}{3}\right) = \frac{1}{3}.$$

Remark: In fact it is true that $f(x) = x$ for every x between 0 and 1 but showing this is much harder. If you would like to try, then one way is to show that $f(q) = q$ for every rational between 0 and 1 with denominator which is a power of 2 and that f is an increasing function. You might want to discuss this with your mentor.

7. Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P and the rays AD and BC meet at Q. The internal angle bisector of $\angle BQA$ meets AC at R and the internal angle bisector of $\angle APD$ meets AD at S. Prove that RS is parallel to CD.
[British Mathematical Olympiad Round 2 2017 Question 3]

SOLUTION

The configuration is shown in the following diagram.



Focusing on the triangle ADC shows that we could use the converse of the intercept theorem on that triangle to show that RS and CD are parallel. Thus we should aim to show that

$$\frac{AS}{SD} = \frac{AR}{RC}. \tag{1}$$

We are given a couple of internal angle bisectors, so start by making use of the angle bisector theorem. The internal angle bisector of $\angle BQA$ meets AC at R. Hence

$$\frac{AR}{RC} = \frac{AQ}{QC}. \tag{2}$$

Also the internal angle bisector of $\angle APD$ meets AD at S. Hence

$$\frac{AS}{SD} = \frac{AP}{PD}. \tag{3}$$

Now $\dfrac{AQ}{QC}$ (see (2)) is a ratio of sides of triangle QCA. We can apply the sine rule to obtain

$$\frac{AQ}{QC} = \frac{\sin \angle QCA}{\sin \angle CAQ}. \tag{4}$$

Similarly in triangle PAD,

$$\frac{AP}{PD} = \frac{\sin \angle ADP}{\sin \angle PAD} = \frac{\sin \angle ADB}{\sin \angle CAD}. \tag{5}$$

Here the second equality is obtained by renaming angles.

We have not yet used any properties of the cyclic quadrilateral, so we seek to apply those. Using angles on the straight line QCB,

$$\sin \angle QCA = \sin(180^\circ - \angle ACB) = \sin \angle ACB = \sin \angle ADB. \tag{6}$$

Here the last equality follows from angles in the same segment. Now $\angle CAQ$ is the same angle as $\angle CAD$, so we can use this equality, along with (6), to substitute in (4). This produces

$$\frac{AQ}{QC} = \frac{\sin \angle ADB}{\sin \angle CAD}. \tag{7}$$

Comparing (5) and (7) shows that

$$\frac{AP}{PD} = \frac{AQ}{QC}. \tag{8}$$

Now substitution from (2) and (3) shows that

$$\frac{AS}{SD} = \frac{AR}{RC}.$$

However, this is the equation (1), which we sought to prove.

You should check that you understand why the equality of these ratios shows that triangles ARS and ACD and that RS is parallel to CD. If necessary, you could discuss it with your mentor.

8. The positive integers m and n are coprime and $m < n$. What is the smallest value of n such that $\dfrac{m}{n}$ has the sequence 251 in its decimal expansion?

SOLUTION

An aspect of this question that is disconcerting is that we are merely told that 251 occurs somewhere in the decimal expansion. So it might be helpful to consider the situation where 251 comes immediately after the decimal point. Thus $\dfrac{m}{n} = 0.251 + a$, where a is a real number whose decimal expansion starts $0.000\ldots$. After multiplying through by 1000 and then multiplying up by n, we have

$$1000m = 251n + an.$$

Note that, by the definition of a, $a < 1$ and so $an < n$. Simplifying the notation by setting $b = an$, we have

$$1000m = 251n + b, \quad (b < n). \tag{*}$$

So now we go through all values of n to find the smallest for which b becomes less than n. Note that, for each n, we pick the smallest m such that $1000m \geqslant 251n$ and then set $b = 1000m - 251n$. Essentially, we are doing a series of long divisions where we produce the first three digits of the quotient all at once. A table (see below) is a good way of displaying the results.

n	m	b
2	1	1000 – 502 = 498
3	1	1000 – 753 = 247
4	2	2000 – 1004 = 996
5	2	2000 – 1255 = 745
6	2	2000 – 1506 = 494
7	2	2000 – 1757 = 243
8	3	3000 – 2008 = 992

It looks as if there is a long way to go without something to guide this exploration. However, in the third column of the table, there are pairs of numbers close together: 498 and 494; 247 and 243. Observe that, as n increases by 4, b decreases by 4. A further feature of the table is that m increases by 1 at each multiple of 4. Do these patterns continue?

First note that $1000m$ has to stay ahead of $251n$. Consequently, since $4 \times 251 = 1004$, m must jump up by at least 1 when n reaches a multiple of 4. Also for reasonably small values of of n, the jump in m will be exactly 1. Suppose we have an n, m pair with $b = 1000m - 251n$. By the preceding remarks, if n increases by 4, then m will go up by 1. Also

$$1000(m + 1) - 251(n + 4) = (1000m - 251n) + (1000 - 1004) = b - 4.$$

Thus, as expected, b has decreased by 4. Exploring some multiples of 4 suggest that we start the process again at $n = 120$. Another table displays the results.

n	m	b
120	31	880
121	31	629
122	31	378
123	31	127
124	32	876
125	32	625
126	32	374
127	32	123

We stop there because the value of b has fallen below the value of n for the first time. We conclude that the required smallest value of n is 127. It can easily be checked that the the decimal expansion of $\dfrac{32}{127}$ starts $0.251\ldots$.

It remains to examine a fraction $\dfrac{m}{n}$, where the 251 sequence does not immediately follow the decimal point in the expansion. Then the decimal expansion has the form $0.*\cdots*251\cdots$, where the asterisks indicate a finite sequence of digits. Then there exists a positive integer k such that

$$10^k\left(\frac{m}{n}\right) = N + 0.251\cdots,$$

where N is an integer. It follows that $10^k m - Nn = (0.251\cdots)n$. An immediate consequence is that $(0.251\cdots)n = c$, say, is an integer. This implies in turn that $\dfrac{c}{n} = (0.251\cdots)$. We can now apply the previous working to the fraction $\dfrac{c}{n}$.

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Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Srinivasa Ramanujan

Sheet 2

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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Version 1.1, Aug 2019

1. A monk begins his ascent of Mount Fuji at sunrise on Monday, reaching the summit by nightfall. He spends that night at the summit and starts down the mountain the following morning, reaching the bottom by dusk on Tuesday. Prove that at some precise time of day, the monk was at exactly the same altitude on Tuesday as he was on Monday.

SOLUTION

The vague definitions of the times the monk starts and finishes his journeys suggest that indicating them by letters may be laborious. Nevertheless, a precise time of day is referred to in the question. The following trick cuts through any difficulties.

Suppose the first monk has a twin brother, an avatar, a fetch, or what you will. On the second day the latter starts his ascent of Mount Fuji at exactly the time that the first monk did the day before. He then, naturally (or supernaturally), repeats the first monk's ascent. The two meet at an exact point on the path. The time of this meeting is the precise time required by the question.

2. Evaluate

$$\int_0^1 \sqrt{x - x^2}\,dx.$$

Comment on any geometrical aspects of this integral.

SOLUTION

When encountering an expression that is nearly a square, it is usually a good idea to complete it. When this is done the integrand becomes

$$\sqrt{\frac{1}{4} - \left(\frac{1}{2} - x\right)^2} = \frac{1}{2}\sqrt{1 - (1-2x)^2} = \frac{1}{2}\sqrt{1 - (2x-1)^2}. \tag{1}$$

This form of the integrand suggests the substitution $2x - 1 = \sin u$. Then $\frac{dx}{du} = \frac{1}{2}\cos u$ and, on making this substitution, the original becomes

$$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 u}\left(\frac{\cos u}{2}\right)du = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 u}\cos u\,du = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2 u\,du.$$

In this final form the integration may be carried out using a double-angle formula and we have

$$I = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2 u\,du = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2u}{2}du = \frac{1}{8}\int_{-\pi/2}^{\pi/2} (1 + \cos 2u)du = \left[u + \frac{\sin 2u}{2}\right]_{-\pi/2}^{\pi/2}.$$

Since $\sin 2u$ is zero at each limit, the integral turns out to be

$$\int_0^1 \sqrt{x - x^2}\,dx = \frac{1}{8}\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right] = \frac{\pi}{8}.$$

So where does geometry come into all this? The progress of the calculations strongly suggests a connection with circles. Returning to (1), make the simpler substitution $2x - 1 = u$ in the integrand. Then $\frac{dx}{du} = \frac{1}{2}$ and the integral I becomes

$$I = \frac{1}{2}\int_{-1}^{1} \sqrt{1 - u^2}\left(\frac{1}{2}\right)du = \frac{1}{4}\int_{-1}^{1} \sqrt{1 - u^2}\,du.$$

However, the last integral is just the area between the upper half of the unit circle centred at the origin and the $x-$axis. This has value $\frac{\pi}{2}$ and so $I = \frac{\pi}{8}$, as before.

Finally, it is worth remarking on a feature of the original integral. The integration is carried out over the interval $[0, 1]$. For all x in this interval, $x \geqslant x^2$. Hence the integrand is well defined, since it is the square root of a non-negative quantity.

3. A sequence of numbers $u_0, u_1, u_2, \ldots$ satisfies

$$u_{n+2} = au_{n+1} + bu_n \qquad (n \geqslant 0),$$

where a and b are real.

a) Find a condition satisfied by a and b when the sequence $u_0, u_1, u_2, \ldots$ is constant. What values can the u_i take?

b) Find conditions satisfied by a and b when the sequence $u_0, u_1, u_2, \ldots$ has period 2. What values can the u_i take?

Solution

a) Suppose that the value of all the terms in the constant sequence is x, then we can re-write the given expression as:

$$x = ax + bx$$

which simplifies to

$$x = (a + b)x$$

This relationship holds either when $x = 0$ or $a + b = 1$ and x can take any real value.

b) We can denote the two alternating terms by x and y, where $x \neq y$. Then the given relation for all $n \geqslant 0$ reduces to one of two forms: either

$$x = ay + bx; \tag{1}$$

or

$$y = ax + by. \tag{2}$$

There are now various cases to consider.
1. If $a = b = 0$, then (1) and (2) together imply that that $x = y = 0$. However, this ***contradicts*** the condition $x \neq y$.
2. If $a = 0$ and $b \neq 0$, then (1) and (2) reduce to $x = bx$, $y = by$, which forces $b = 1$. It follows that x and y can take any distinct real values.
3. If $a \neq 0$, then making x the subject in (2) yields $x = \dfrac{(1 - b)y}{a}$. Substituting for x in (1) gives us

$$\frac{(1 - b)y}{a} = ay + \frac{b(1 - b)y}{a}.$$

Multiplying up and rearranging produces $(1 - b)^2 y = a^2 y$. There are now two cases: either (i) $y = 0$ or (ii) $(1 - b)^2 = a^2$.
(i) Substituting $y = 0$ in (2), we have $ax = 0$ and, since $a \neq 0$, this in turn implies $x = 0$. Again this ***contradicts*** the condition $x \neq y$.
(ii) The condition $(1 - b)^2 y = a^2 y$ means that $1 - b = a$ or $1 - b = -a$. Rearranging, we have

the two sub-cases $a + b = 1$ and $b - a = 1$.
If $a + b = 1$, then substitution in (1) produces

$$x = (a + b)x = ay + bx \qquad \text{or} \qquad ax = ay.$$

But we are assuming $a \neq 0$ so this again ***contradicts*** the condition $x \neq y$.
If $b - a = 1$, then substitution in (1) produces

$$x = (b - a)x = ay + bx \qquad \text{or} \qquad -ax = ay.$$

Again we can cancel to obtain $-x = y$. This will produce a sequence of period two. It will have the form $u, -u, u, -u. \ldots$, where u can take any non-zero real value.

4. Imagine that you have a set of 16 distinct weights and another set of 15 distinct weights. They belong to different systems and you do not know how to convert from one system to the other. However you have checked that the union of the two sets of weights constitutes a set of 31 distinct weights. You wish to know which of the 31 weights ranks as the middle one. In their respective systems the weights are labelled in order. For example, in the first set the weights are labelled from 1 (the lightest) to 16 (the heaviest); the other set is similarly labelled. You also have a balance to compare weights. Explain how you can efficiently find the middle ranking weight.

Solution

Suppose the weights in the first set are denoted by $I_1, I_2, \ldots, I_{16}$ and the weights in the second set are denoted by $J_1, J_2, \ldots, J_{15}$. Further suppose that I_1 is the lightest and then the weights are ranked in increasing order up to I_{16}. Similar notation is used for the second set up to J_{15}. An efficient overall strategy is as follows: pair up ranks from each set so that their sum is 17.

We can initially weigh the pair I_9 and J_8. If $I_9 > J_8$ then we know that all weights $I_9, \ldots, I_{16}$ will be ranked higher than the 16th place (the middle position) in the combined order. In this context, ensure you understand why this last statement is true. You could try asking yourself if I_9 could be in the 16th place or below the 16th place. Because the 8 weights $I_9, \ldots, I_{16}$ are strictly above the middle position, we can remove them. If on the other hand, $I_9 < J_8$, then by a similar argument, we can remove $J_8, \ldots, J_{15}$.

At this stage, each group has at least 7 weights remaining. Again, an appropriate choice of an I_i and an J_j where $i + j = 17$ will enable us to remove another 4 weights. At this stage the smaller group would have at least 3 weights. By choosing the right pair to compare we can remove another 2. Note that this pair consists of the second highest remaining weights from each group. At this stage we have removed $8 + 4 + 2 = 14$ weights. There are 17 weights left.

Now we have at least two more comparisons to make. Of the remaining 17 weights, compare the highest ranked one from each set: this will give us the 17th weight in the combined system. On repeating the process, we will have the 16th one. If out of the last 17 weights J_1 gets removed, then we do not need the final weighing.

Remark: You might like to consider whether the method given is the most efficient or whether the middle position could be found with fewer comparisons.

5. *a*) Find a formula for the number of subsets of a set with n elements, where n is a non-negative integer. You should justify your answer.

b) Prove that, among $n + 1$ integers from the set $\{1, \ldots, 2n\}$, there are two which are coprime.

c) From ten distinct two-digit integers, one can always choose two disjoint, non-empty subsets whose elements have the same sum. Prove this statement.

SOLUTION

a) Suppose we are constructing a subset of the set $\{a_1, a_2, \ldots, a_n\}$. Imagine that we have n slots and that we go through the elements of the set. For each a_i we decide whether to drop it in the ith slot (marking the decision by I for *In*) or to leave the ith slot empty (marking this decision by O for *Out*). Thus the whole subset can be represented by a string of I's and O's, of string length n. The number of possible strings of this type is 2^n and so the total number of subsets is also 2^n. *(See Remark 1 below.)*

b) Among $n + 1$ integers from $1, \ldots, 2n$ there are two consecutive integers. They must be coprime.

c) Form a set S, say, of ten two-digit integers. They are all less than or equal to 99. By the answer to part *a)*, S has $2^{10} = 1024$ subsets. The sum of the numbers in any subset of S is less than or equal to $10 \times 99 = 990$. Since the number of possible sums is less than the number of subsets, it follows that there must be at least two distinct subsets with the same sum. If two of these subsets are disjoint, then we are done. If not, we remove the common elements to obtain two subsets with the required property.

Remark 1: Note that replacing I by 1 *and O by* 0 *in any string produces a binary number with n digits. (Food for thought?) Again, another approach to part a) of the question uses the binomial theorem. You may want to explore this or discuss it with your mentor.*

Remark 2: This is a simple example of the pigeonhole principle. It states that if you have $kn + 1$ *pigeons contained in n pigeonholes then there must be some pigeonhole containing* $k + 1$ *pigeons (or more). In this case, the subsets are the pigeons and the possible sums (which do not number more than 990) are the pigeonholes.*

6. *a*) A class of sixth-formers is given a maths test. Of the questions, $\frac{2}{3}$ are deemed to be hard. It is reported that each such problem was not solved by $\frac{2}{3}$ of the students. It is also reported that at least $\frac{2}{3}$ of the students passed the test and that each student who passed solved at least $\frac{2}{3}$ of the problems set.
Is the situation described possible?

b) Answer the question posed in *a)* but with $\frac{2}{3}$ replaced by $\frac{3}{4}$.

c) Answer the question posed in *a)* but with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

SOLUTION

In this solution we use the following definitions. Let:
the number of students be denoted by N;
the number of problems be denoted by Q;

the number of students who pass be denoted by P;
the number of hard problems be denoted by H;
the fraction in the relevant part of the question be denoted by r.
Note that, according to these definitions, a problem is deemed hard if it was not solved by at least rN students. Also a student who passes has solved at least rM problems.

a) Here $r = \frac{2}{3}$. This scenario is possible and can be demonstrated in the case of a small (presumably advanced) class of three students S_1, S_2 and S_2. They sit a (presumably advanced) paper of three problems Q_1, Q_2 and Q_3. Suppose that S_1 solves Q_1 and Q_3; S_2 solves Q_2 and Q_3 but S_3 solves neither Q_1 nor Q_2. Check carefully for yourself that the given conditions are satisfied. Note that S_1 and S_2 passed and that Q_1, Q_2 are the hard problems.

b) Now $r = \frac{3}{4}$. and there are more choices to investigate. The data could be organised into an $N \times M$ table with students who pass and the hard questions in the top left-hand corner. The entries in the table will consist of a plus sign where a student has solved a problem and a minus sign otherwise. As an illustration, the table below demonstrates the simple case in *a)*.

	S_1	S_2	S_3
Q_1	+	−	−
Q_2	−	+	−
Q_3	+	+	?

We consider the number of plus and minus signs in the table: denote these numbers by K and L respectively. We have the following inequalities.

$$K \geqslant P \times rM \geqslant rN \times rM = r^2MN.$$

$$L \geqslant H \times rN \geqslant rM \times rN = r^2MN.$$

Adding these inequalities gives $MN = K + L \geqslant 2r^2MN = 2 \times \frac{9}{16}MN = \frac{9}{8}MN$, which is a ***contradiction.*** Hence the situation is impossible.

c) For the last part, we might try the same approach again but unfortunately it will not work this time because now $2r^2 = 2 \times \frac{16}{49} = \frac{32}{49} \leqslant 1$. This suggests that we should consider a smaller portion of the table and the obvious choice is the top left-hand corner involving the students who passed and the hard problems. (In the illustrative example, the relevant cells are shaded.) Further, suppose that K and L now denote the number of plus / minus signs in the sub-table.

Then K must be greater than or equal to the number of hard problems solved by students who passed. This is $P \times \frac{4}{7}H = \frac{4}{7}PH$. Again L must must be greater than or equal to problems not solved by by students who passed. This is $H \times \frac{4}{7}P = \frac{4}{7}PH$. Adding the inequalities corresponding to these two statements yields

$$PH = K + L \geqslant \frac{8}{7}PH.$$

This impossible inequality furnishes the required ***contradiction*** in this case.

7. *Introductory note: In triangle ABC, the median from A is the line AM, where M is the midpoint of the side BC. In any triangle the medians intersect at the point called the centroid, which divides each median in the ratio 2:1.*
In the convex quadrilateral $ABCD$, the points A', B', C' and D' are the centroids of the triangles BCD, CDA, DAB and ABC, respectively.

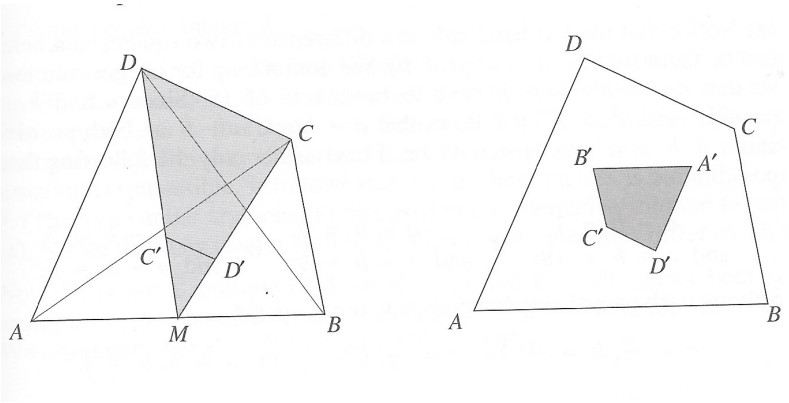
a) By considering the triangle MCD, where M is the midpoint of AB, prove that $C'D'$ is parallel to DC and that $C'D' = \frac{1}{3}DC$.

b) Prove that the quadrilaterals $ABCD$ and $A'B'C'D'$ are similar.

[Mathematical Olympiad for Girls 2013 Question 2]

SOLUTION

The diagrams below serve to illustrate the two parts of the question.



a) We are given that M is the midpoint of AB, so CM is a median of triangle ABC. Since D' is the centroid of triangle ABC, it lies on CM and $CD' : D'M = 2 : 1$. Similarly $DC' : C'M = 2 : 1$. This means that triangles MCD and $MC'D'$ have two pairs of sides in the same ratio, namely 3:1. Furthermore the angle CMD is common to both triangles. Hence the triangles are similar. *(See Remark 1 below.)*

It follows that the third sides are also in the ratio 3:1, so $C'D' = \frac{1}{3}CD$,as required. Furthermore, corresponding angles in the two triangles are equal, so $\angle MD'C' = \angle MCD$. We conclude that $C'D'$ and DC are parallel.

b) First recall that, for two quadrilaterals to be similar, then all pairs of corresponding sides must be in the same ratio ***and*** all pairs of corresponding angles must be equal. There are no shortcuts as in the case of triangles. (Make sure you understand why this is so: for instance, try drawing two non-similar quadrilaterals where all pairs of corresponding angles are equal.)

In part *a)* we saw that $C'D'$ is parallel to DC. Similarly, $D'A'$ is parallel to AD. Hence $\angle A'D'C' = \angle ADC$. It can be shown analogously that the other corresponding angles in the two quadrilaterals are equal. It can also be shown (as in part *a)*) that the lengths of corresponding sides of the two quadrilaterals are all in the ratio 3:1. Hence the two quadrilaterals are similar.

Remark 1: If you want to clarify your ideas about similarity, then you could discuss them with your mentor. Or, if you did Hanna Neumann, Sheet 3, you could look back to the discussion of similar triangles in Question 4. This also links the various criteria to the corresponding ones for congruence. Again, if you have access to the UKMT Pathways publication Crossing the Bridge,

then you will find material pertaining to this question in Chapter 5.
Remark 2: The results in this question still hold when $ABCD$ is not convex. Indeed they still hold even if $ABCD$ is self-intersecting.

8. Suppose that every point in the plane is coloured either red or blue.

a) Must there be two points of the same colour at distance 1 from each other?

b) Must there be one of

- two red points at distance 1 from each other, or
- three equally spaced blue points forming a line of length 2?

SOLUTION

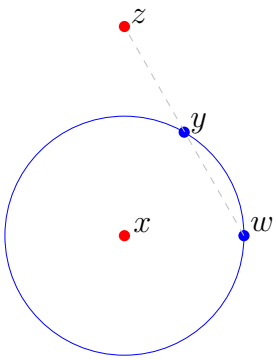
(a) Consider the vertices of an equilateral triangle of side length 1. If we colour these three vertices red and blue, there must be two of them that are the same colour.

(b) First we introduce some notation: let L denote three points in a line where consecutive points are distance 1 apart. We will show that it is always possible to find either two red points at distance 1 or a blue copy of L.

Suppose that there is some red-blue colouring where there is neither two red points distance 1 apart nor a blue copy of L. We aim for a contradiction.

If every point were coloured blue then there would be a blue copy of L. So there must be some red point, call it x. Consider the circle C_1 which has radius 1 with centre x. If there were any red points on this circle, there would be two red points at distance 1, so every point on C_1 must be blue.

Let w and y be two points on C_1 that are distance 1 apart. If z is such that w, y, z is a copy of L, then since w and y are blue, z must be red. (Make sure you understand why z blue leads directly to a contradiction.)



Rotating w and y around the circle C_1 we can see that by symmetry z must trace the locus of a circle, C_2. Thus every point on the circle C_2 must be red. We can find two points on the circle C_2 that are distance 1 from each other. Thus there are two red points at distance 1 from each other and we have a contradiction. Thus the desired result is proved.

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Mentoring Scheme

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Srinivasa Ramanujan

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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Version 2.0, Nov 2020

1. You have a set of n positive integers, where $n \geqslant 5$. Prove that the set must contain either a multiple of 5 or two integers whose difference is divisible by 5.

SOLUTION

Suppose that the set does not contain a multiple of 5. Then the existence of the other property can be confirmed by a straightforward application of the pigeonhole principle. This was last discussed in the solution to Ramanujan Sheet 2, Question 5. Because of our assumption, any element of the set must leave remainder 1, 2, 3 or 4 on division by 5. Label each of four pigeonholes by these remainders. The pigeonhole principle tells us that there must be at least two integers in at least one of the pigeonholes. Pick such a pair of integers: since they have the same remainder on division by 5, their difference must be divisible by 5.

2. Does $z^2 = 6x^2 + 2y^2$ have a solution in integers x, y, z not all zero? If you think so give all possible solutions. If you think not, then prove it.

SOLUTION

Without loss of generality we can say that, if such a solution does exist, then there must be one where x, y, z have no common factors. (Stop and check that you understand why the form of the equation dictates this.) Rewrite the equation as

$$6x^2 = z^2 - 2y^2. \tag{1}$$

Then since the left-hand side is divisible by 3, the same must be true of $z^2 - 2y^2$.

In order to analyse this more carefully we re-write z and y in quotient-remainder form. Say

$$y = 3u + l \quad \text{and} \quad z = 3v + m,$$

where $0 \leqslant l \leqslant 2$ and $0 \leqslant m \leqslant 2$. Substituting into (1) yields

$$6x^2 = (3u + l)^2 - 2(3v + m)^2 = 3(3u^2 + 2ul - 3v^2 - 2vm) + (l^2 - 2m^2).$$

Here the only terms which are not obviously divisible by 3 are those in the bracket $(l^2 - 2m^2)$.

Now is the time to review arithmetic modulo 3. Check for yourself that all squares are congruent to 0 or 1 modulo 3. (For example $(3q+2)^2 = 3(3q^2+4q)+4 = 3(3q^2+4q+1)+1 \equiv 1 \pmod 3$.) The table below shows the relevant calculations modulo 3, which involve the possible values of l^2 and m^2.

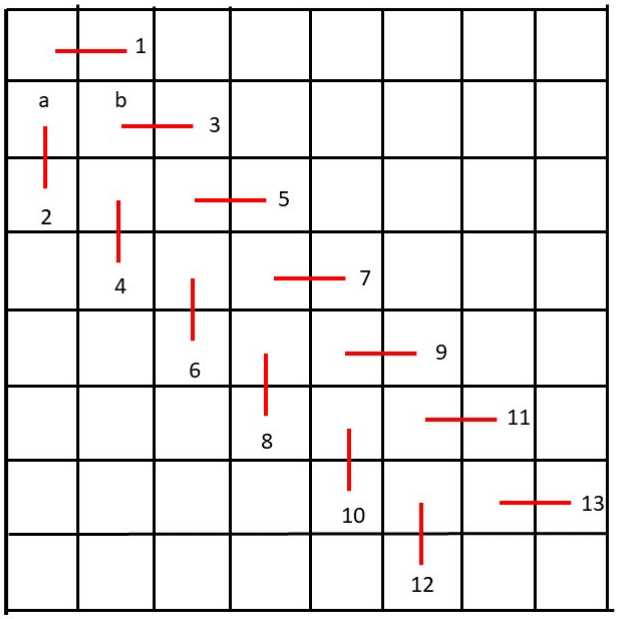
l^2	m^2	$l^2 - 2m^2$
0	0	0
0	1	1
1	0	1
1	1	2

The table shows that, out of all possible combinations of l and m, it is only when they are both congruent to 0, that $l^2 - 2m^2$ is divisible by 3. This in turn implies that y and z are both multiples of 3. However, when this is true, the right-hand side of (1) has a factor of 9. Back to the left-hand side of (1): this must have a factor of 9 as well. The final step is to recognise that this means x must have a factor of 3. This ***contradicts*** our previous assumption that x, y and z have no common factor and the required result is proved.

3. Is it possible to tile an 8×8 chessboard with dominoes so that no two dominoes form a 2×2 square?
Note: A domino is a 2×1 tile. A tiling of the board requires that every square of the board be covered by a domino, with no overlaps.

SOLUTION

The diagram below can be referred to as you check through the solution. In it the red bars indicate dominoes, each is given a number to its right. The small letters refer to squares on the underlying board.



Because there are so many possibilities, it is worth seeing whether a contradiction argument will furnish a result. Suppose that such a tiling exists. Without loss of generality, we may start at the top left-hand (unlettered) corner of the board. This has to be covered by a domino: again without loss of generality suppose that it is placed horizontally. It is represented by domino 1 in the diagram. Now consider square a. It cannot be covered by a horizontal domino because ——. (Fill in the reason for yourself.) Hence the domino covering square a must be the vertical domino 2. Now move on to square b: this cannot be covered by a vertical domino because ——. (Again fill in the reason for yourself.) Hence the domino covering square b must be the horizontal domino 3.

Proceeding in this way, you will find that you are forced to position dominoes 4 to 13 in the manner indicated on the diagram. Further, you are then forced to cover the two squares in the bottom right-hand corner by a horizontal domino. But this produces a ***contradiction,*** in the form of a square with two horizontal dominoes in the bottom right-hand corner. Thus it is not possible to tile the board with dominoes satisfying the given condition.

4. This question explores ***unit fractions***, which are fractions with numerator 1.

a) Express each of $\frac{2}{3}$, $\frac{11}{13}$ as a finite sum of distinct unit fractions.

b) Leonardo of Pisa (aka Fibonacci, about 1170 to 1240 CE) devised an algorithm to express any fraction $\frac{m}{n}$, with $0 < \frac{m}{n} < 1$, as a finite sum of distinct unit fractions. His method starts by finding the largest unit fraction $\frac{1}{u_1}$, say, that is less than or equal to $\frac{m}{n}$. It is then possible to write

$$\frac{m}{n} = \frac{1}{u_1} + \left(\frac{m}{n} - \frac{1}{u_1}\right). \qquad (*)$$

Call the fraction in brackets in (*) the ***remainder fraction.*** If this remainder fraction is not a unit fraction, then repeat the process on it. Continue to repeat as many times as necessary.
Use Fibonacci's algorithm to express $\frac{4}{13}$ as a sum of distinct unit fractions.

c) Show that this remainder fraction has a smaller numerator than $\frac{m}{n}$.

d) Explain why Fibonacci's algorithm always produces a sum of the desired type.

SOLUTION

a) The first expression is familiar; try to find a sum that produces the second fraction. Possible answers are $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ and $\frac{11}{13} = \frac{1}{2} + \frac{1}{3} + \frac{1}{78}$.

b) Applying the algorithm,

$$\frac{4}{13} = \frac{1}{4} + \left(\frac{4}{13} - \frac{1}{4}\right) = \frac{1}{4} + \frac{3}{52} = \frac{1}{4} + \frac{1}{18} + \left(\frac{3}{52} - \frac{1}{18}\right) = \frac{1}{4} + \frac{1}{18} + \left(\frac{54-52}{52\times 18}\right).$$

Thus

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{52\times 9} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

c) First we have to identify the "largest unit fraction $\frac{1}{u_1}$, say, that is less than or equal to $\frac{m}{n}$". This rather vague aim can be pinned down by recognising that u_1 must be the smallest integer greater than or equal to $\frac{n}{m}$*. Pause for a moment to make sure you understand why this is so. You could ask yourself: what would happen if there were another integer $u' \neq u_1$, which could produce the required unit fraction.
**In this context, see Note I below.*

In the example in part *b)*, you may have noticed that the numerators of the remainder fractions decreased at each step. We shall see whether this is true in general because, if it is, then the process must terminate. To this end, consider the first two fractions. We are given that $0 < \frac{m}{n} < 1$ and so $\frac{n}{m} > 1$. Consequently $u_1 \geqslant 2$, which implies that $u_1 - 1 \geqslant 1 > 0$. We also have $u_1 > \frac{n}{m}$ because otherwise $\frac{n}{m}$ would be an integer and $\frac{m}{n}$ a unit fraction already.

Summing up,

$$u_1 > \frac{n}{m} > u_1 - 1 > 0. \qquad (1)$$

Note that the second of these inequalities must be true because $u_1 - 1 \geqslant \frac{n}{m}$ would contradict the definition of u_1. Because the terms in (1) are positive, it follows that

$$\frac{1}{u_1 - 1} > \frac{m}{n} > \frac{1}{u_1}. \tag{2}$$

Now consider the first inequality in (2): it implies that $n > m(u_1 - 1)$, which in turn implies that

$$m > mu_1 - n. \tag{3}$$

But $mu_1 - n$ is the numerator of the remainder fraction in (*) and m is the numerator of the original fraction, so the required result is proved. This argument can be repeated as many times as is necessary before the process terminates.

d) You may also have observed that the denominators of the unit fractions created by the algorithm increased at each step. Let us now explore that observation. The counterpart to (*) at the next stage is

$$\frac{m}{n} - \frac{1}{u_1} = \frac{mu_1 - n}{nu_1} = \frac{1}{u_2} + \left(\frac{mu_1 - n}{nu_1} - \frac{1}{u_2}\right), \tag{4}$$

where u_2 is the smallest integer greater than or equal to $\dfrac{nu_1}{mu_1 - n}$. By (3) $mu_1 - n < m < n$ so

$$\frac{nu_1}{mu_1 - n} > \frac{nu_1}{n} = u_1. \tag{5}$$

Because this inequality is strict, the smallest integer greater than or equal to $\dfrac{nu_1}{mu_1 - n}$ must also be greater than u_1. But this integer is u_2 and so $u_2 > u_1$. This is the result we wanted to verify. Finally note that the fact these denominators are strictly increasing ensures that all the unit fractions produced by the algorithm are distinct.

Note 1: The function which associates with a real number x the smallest integer greater than or equal to x is called the ***ceiling function.*** *The ceiling of x is denoted by $\lceil x \rceil$. You may have met the comparable* ***floor function****, $\lfloor x \rfloor$. The floor of a real number x is the largest integer which is less than or equal to x.*
Note 2: It is straightforward to verify that the representation of a fraction given by Fibonacci's algorithm is not necessarily unique. Moreover, applying it can involve a large number of steps and other algorithms exist.
Note 3: In general, the ancient inhabitants of Egypt used unit fractions only: hence their alternative name of Egyptian fractions. Many surviving inscriptions demonstrate their use in practical calculations. However, $\frac{2}{3}$ is often used in non-unit form and the same is true, to a lesser extent, of $\frac{3}{4}$.

5. Let x and y be real numbers and let

$$p = x^2 + xy + y^2. \tag{1}$$

Also let

$$q = x^2 - xy + y^2. \tag{2}$$

a) Show that $0 \leqslant q \leqslant 3p \leqslant 9q$.

b) Show that if p and q are real numbers satisfying the inequalities in *a)*, then there exist real numbers x and y such that the relations (1) and (2) hold.

SOLUTION

a) To show that $0 \leqslant q \leqslant 3p \leqslant 9q$, we need to establish the three inequalities $0 \leqslant q$, $q \leqslant 3p$ and $3p \leqslant 9q$ (this last one is the same as $p \leqslant 3q$).

Consider $q \leqslant 3p$: by(1) and (2), we need to show that $x^2 - xy + y^2 \leqslant 3(x^2 + xy + y^2)$. Rearranging and dividing by 2, it looks as though we need to show that $x^2 + 2xy + y^2 \geqslant 0$. But this is true because the left-hand side is $(x + y)^2$. Similarly, $p \leqslant 3q$ reduces to $(x - y)^2$. Although starting with what you need to prove and working backwards (as we have done here) is often a good way to see how you might prove an inequality, it does not constitute a valid proof.

One way to lay out the proof is as follows. Adding, then subtracting, (1) and (2) gives

$$p + q = 2(x^2 + y^2) \tag{3}$$

and

$$p - q = 2xy. \tag{4}$$

Now, as squares are non-negative,

$$0 \leqslant (x + y)^2 = \frac{p + q}{2} + p - q = \frac{3p - q}{2} \tag{5}$$

and

$$0 \leqslant (x - y)^2 = \frac{p + q}{2} - p + q = \frac{3q - p}{2} \tag{6}$$

Hence by (5), $q \leqslant 3p$ and by (6), $p \leqslant 3q$. Combining these, we have $q \leqslant 3p \leqslant 9q$. Now we just need to prove that $0 \leqslant q$. But we have $q \leqslant 9q$ so $0 \leqslant 8q$ and we are done.

b) Now we assume that the inequalities in part *a)* are satisfied and we have to show that there exist x and y for which (1) and (2) hold .

We aim to write x and y explicitly in terms of p and q, using our work in part *a)* as a guide. Then we will check that (1) and (2) hold for this x and y. The relations in (5) and (6) allow us to do exactly this. We would like to take

$$x + y = \sqrt{\frac{3p - q}{2}} \qquad \text{and} \qquad x - y = \sqrt{\frac{3q - p}{2}},$$

so

$$x = \frac{1}{2}\left(\sqrt{\frac{3p - q}{2}} + \sqrt{\frac{3q - p}{2}}\right) \qquad \text{and} \qquad y = \frac{1}{2}\left(\sqrt{\frac{3p - q}{2}} - \sqrt{\frac{3q - p}{2}}\right)$$

Note that this is possible since the inequalities in part *a)* tell us that $3p - q \geqslant 0$ and $3q - p \geqslant 0$. We must now just check that this x and y do actually satisfy (1) and (2). We can do this simply by brute force or with a bit more finesse as follows. We have $xy = \dfrac{p - q}{2}$, using the difference of two squares. Also $x^2 + y^2 = \dfrac{p + q}{2}$, using the identity $(a - b)^2 + (a + b)^2 = 2a^2 + 2b^2$. From these (1) and (2) follow immediately.

6. Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$. Also, for $n \geqslant 2$, define $f_n(x) = f_1(f_{n-1}(x))$. Find the value of x that satisfies $f_{1001}(x) = x - 3$.

Solution

In a question like this one has to start with the first few terms in the sequence, if only because there is not anything else one can do. So we start by simplifying $f_1(x)$ to obtain

$$f_1(x) = \frac{2}{3} - \frac{3}{3x+1} = \frac{6x-7}{3(3x+1)}.$$

Moving on,

$$f_2(x) = f_1(f_1(x) = \frac{6f_1(x)-7}{3(3f_1(x)+1)} = \frac{6\left(\frac{6x-7}{3(3x+1)}\right)-7}{3\left(\frac{3(6x-7)}{3(3x+1)}+1\right)} = \frac{2(6x-7)-7(3x+1)}{3(6x-7+3x+1)}.$$

A final simplification gives

$$f_2(x) = \frac{-9x-21}{3(9x-6)} = -\frac{3x+7}{3(3x-2)}.$$

It follows that

$$f_3(x) = f_1(f_2(x)) = \frac{6f_2(x)-7}{3(3f_2(x)+1)} = \frac{6\left(-\frac{3x+7}{3(3x-2)}\right)-7}{3\left(3\left(-\frac{3x+7}{3(3x-2)}\right)+1\right)} = \frac{6\left(\frac{3x+7}{3(3x-2)}\right)+7}{3\left(3\left(\frac{3x+7}{3(3x-2)}\right)-1\right)}.$$

Further simplification yields

$$f_3(x) = \frac{2\left(\frac{3x+7}{(3x-2)}\right)+7}{3\left(\frac{3x+7}{(3x-2)}-1\right)} = \frac{2(3x+7)+7(3x-2)}{3(3x+7-3x+2)} = \frac{27x}{3\times 9} = x.$$

This last equation shows that the sequence of functions has period 3 and therefore $f_{999}(x) = x$. Furthermore

$$f_{1001}(x) = f_2(x) = -\frac{3x+7}{3(3x-2)}.$$

Equating this to $x-3$ and multiplying up yields

$$-(3x+7) = 3(3x-2)(x-3).$$

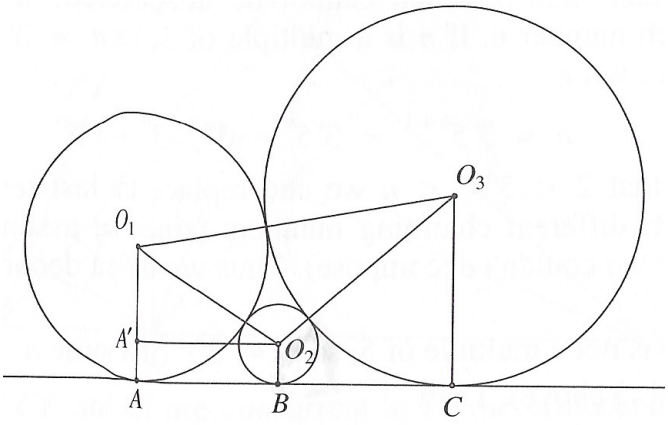
Rearranging, we have the quadratic equation $9x^2 - 30x + 25 = 0$. But this is just $(3x-5)^2 = 0$ and so the required solution is $x = \frac{5}{3}$.

7. Circles of radius r_1, r_2 and r_3 touch each other externally. They also touch a common tangent at points A, B and C externally, where B lies between A and C. Prove that $16(r_1 + r_2 + r_3) \geqslant 9(AB + BC + CA)$.

[British Mathematical Olympiad Round 2 2016 Question 1]

Solution

The configuration is shown in the diagram below.



In this diagram, the centres of the circles are denoted by O_1, O_2 and O_3 as shown. Let A' be the foot of the perpendicular from O_2 to AO_1. Since $A'O_2BA$ is a rectangle, $A'O_2 = AB$. Also $O_1A' = r_1 - r_2$ and $O_1O_2 = r_1 + r_2$. Hence by Pythagoras's theorem on triangle $O_1A'O_2$,

$$AB^2 = A'O_2^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2.$$

Similarly,

$$AC^2 = 4r_1r_3 \quad \text{and} \quad BC^2 = 4r_2r_3.$$

Furthermore, $AB + BC = AC$, so $\sqrt{r_1r_3} = \sqrt{r_1r_2} + \sqrt{r_2r_3}$. Hence, if we set $a = \sqrt{r_1}$ and $b = \sqrt{r_3}$, we learn that

$$\sqrt{r_2} = \frac{\sqrt{r_1r_3}}{\sqrt{r_1} + \sqrt{r_3}} = \frac{ab}{a+b}.$$

Thus we are required to prove the following inequality:

$$16\left(a^2 + b^2 + \left(\frac{ab}{a+b}\right)^2\right) \geqslant 9\left(2ab + \frac{2a^2b}{a+b} + \frac{2ab^2}{a+b}\right) = 18\left(ab + \frac{ab(a+b)}{a+b}\right) = 36ab.$$

This amounts to verifying that

$$4((a^2 + b^2)(a+b)^2 + a^2b^2) \geqslant 9ab(a+b)^2.$$

But this is equivalent to the requirement that $4((a^2 + b^2)(a + b)^2 + a^2b^2) - 9ab(a + b)^2$ be non-negative. However,

$$\begin{aligned}
4((a^2 + b^2)(a + b)^2 + a^2b^2) - 9ab(a + b)^2 &= 4(a + b)^2(a - b)^2 - ab(a + b)^2 + 4a^2b^2 \\
&= 4(a + b)^2(a - b)^2 - ab(a^2 + 2ab + b^2 - 4ab) \\
&= 4(a + b)^2(a - b)^2 - ab(a - b)^2) \\
&= (a - b)^2(4a^2 + 8ab + 4b^2 - ab) \\
&= (a - b)^2(4a^2 + 7ab + 4b^2)
\end{aligned}$$

This last expression is clearly non-negative and so the required result is proved.

8. ***Exploring necklace design***

To relax from the rigours of setting problems and baking, Mrs Logistic makes necklaces. Each necklace contains eight beads strung onto a cord, the beads being either black or white. She normally starts with a basic set of distinct patterns, each of which contains eight beads. Within each pattern each bead is assigned a position from 1 to 8. Note that, for a set to be a basic set, it must have the property that no two patterns in it will ***combine*** together to produce another existing pattern in the set. Here ***combine*** has a specific meaning, which is explained immediately below.

Combination rule: Take two necklaces from the basic set and compare the two beads in each assigned position. If these beads are of the same colour, then the new necklace has a white bead in that position. If the two beads in the same position are of different colours, then the new necklace has a black bead. Successive combinations of necklaces are possible, and Mrs L is allowed to combine a basic pattern with itself.

a) Suppose that Mrs L has n distinct patterns in the basic set and in none of them are all the beads white.
How many different patterns, including the basic set, can she generate?

b) Now let k be an integer with $0 \leqslant k \leqslant n$. Further, suppose that Mrs L has k patterns in the basic set with odd numbers of black and white beads. When Mrs L has made all possible patterns from her basic set, what is the proportion of necklaces with odd numbers of black and white beads?

SOLUTION

a) To get a feel for this problem, you are strongly recommended to try out some cases for small values of n. This should suggest to you that n basic necklaces will give rise to 2^n distinct patterns overall. However, we need to be sure that this result continues to hold for all values of n and, in particular, that all the 2^n patterns we think are produced at each stage are distinct. To pin down this result we set up some algebraic notation to describe a typical necklace. We use the "$\oplus$" symbol to denote combination. Precisely, for any two necklaces p_1 and p_2, $p_1 \oplus p_2$ is the combination of p_1 and p_2. Furthermore, we know that $p_1 \oplus p_2$ is another permitted necklace. It is easy to check that the following properties are consequences of the combination rule. (Do the properties remind you of another algebraic situation?)

- The all-white necklace w is an identity element, because $p \oplus w = p$ for any necklace p.
- All possible necklaces are self inverse, since $p \oplus p = w$.
- $\oplus$ is commutative, that is, $p_1 \oplus p_2 = p_2 \oplus p_1$.
- $\oplus$ is associative, meaning that $(p_1 \oplus p_2) \oplus p_3 = p_1 \oplus (p_2 \oplus p_3)$.

The foregoing properties imply that, unless we want to create the all-white necklace w, we never have to use a basic necklace twice or more. For example, $p_1 \oplus p_2 \oplus p_1 = p_1 \oplus p_1 \oplus p_2 = w \oplus p_2 = p_2$.

Furthermore, if A is a set of basic necklaces, then each pattern we can produce is associated with a subset of A. If $A = \{p_1, p_2, \ldots, p_n\}$, then a typical necklace N can be expressed as

$$N = x_1 p_1 \oplus x_2 p_2 \oplus \cdots \oplus x_n p_n. \tag{1}$$

where $x_i \in \{0, 1\}$. Specifically, if $x_i = 1$, then the term in p_i appears in N; if $x_i = 0$, then the term in p_i does not appear in N. Hence we can produce at most 2^n patterns from a basic set of

size n. It can also be observed when the given length of the necklaces is eight, then $n \leq 8$. This is because each of the eight positions in N can contain either a black bead or a white bead. Hence the total of possible distinct black and white patterns is 2^8. In other words, we cannot create more than 2^8 many distinct necklaces when we have eight beads per necklace.

We now show that the number of patterns we can produce is *exactly* 2^n. We will argue by contradiction. If the number of patterns we can produce is smaller than 2^n then there must be two sets $B = \{b_1, b_2, \ldots, b_{k_b}\}$, $C = \{c_1, c_2 \ldots, c_{k_c}\}$ such that $B, C \subseteq A$, $B \neq C$, $k_b \leq n$, $k_c \leq n$ and

$$b_1 \oplus b_2 \oplus \ldots \oplus b_{k_b} = c_1 \oplus c_2 \oplus \ldots \oplus c_{k_c}. \tag{2}$$

Since $B \neq C$, there is either a necklace in B that is not in C, or there is a necklace in C that is not in B. Without loss of generality, assume $b_1 \notin C$. We combine both sides of Equation 2 with $b_2 \oplus \ldots \oplus b_{k_b}$ to get the following chain, where each statement follows from the preceding one.

$$\begin{aligned} b_1 \oplus b_2 \oplus \ldots \oplus b_{k_b} \oplus b_2 \oplus \ldots \oplus b_{k_b} &= c_1 \oplus c_2 \oplus \ldots \oplus c_{k_c} \oplus b_2 \oplus \ldots \oplus b_{k_b} \\ b_1 \oplus b_2 \oplus b_2 \ldots b_{k_b} \oplus b_{k_b} &= c_1 \oplus c_2 \oplus \ldots \oplus b_2 \oplus \ldots \oplus b_{k_b} \qquad (3) \\ b_1 \oplus w \ldots \oplus w &= c_1 \oplus c_2 \oplus \ldots \oplus b_2 \oplus \ldots \oplus b_{k_b} \qquad (4) \\ b_1 &= c_1 \oplus c_2 \oplus \ldots \oplus b_2 \oplus \ldots \oplus b_{k_b} \end{aligned}$$

Note that these manipulations follow from the properties of $\oplus$. In particular, the commutative and associative rules are key. The associative law means that, when we are combining a string of terms, we can tackle the calculations in any convenient order. The commutative property enables us to alter the order of any two adjacent terms without affecting the way they combine. (If these are unfamiliar ideas, then try a similar calculation for a small value of n, putting in all the details.) Now recall that all the elements of B are distinct and b_1 is not in C. We have shown we can produce b_1 by combining other necklaces in A. This contradicts the definition of A as a basic set. We conclude that each combination is unique, apart from the combination of any two identical necklaces, which produces w. Therefore there are exactly 2^n distinct patterns produced when the given basic set has n patterns and $n \leq 8$.

b) Suppose that p_1 has d_1 black beads and p_2 has d_2 black beads. In addition, suppose there are d_{12} positions that are black in both p_1 and p_2. Positions that are black in both p_1 and p_2, as well as positions that are white in both p_1 and p_2, become white in $p_1 \oplus p_2$. Hence the number of positions that are black in $p_1 \oplus p_2$ is $d_1 - d_{12} + d_2 - d_{12} = d_1 + d_2 - 2d_{12}$. This number is odd if and only if d_1 and d_2 have opposite parity (either odd-even or even-odd).

This means if we call a necklace with an odd number of black beads an odd necklace and a necklace with an even number of black beads an even necklace, then

$$\text{odd necklace} \oplus \text{odd necklace} = \text{even necklace}$$

$$\text{even necklace} \oplus \text{even necklace} = \text{even necklace}$$

$$\text{odd necklace} \oplus \text{even necklace} = \text{odd necklace}$$

Suppose the given basic set has k_{odd} odd necklaces and k_{even} even necklaces. (This means that, in the notation of the question, $k = k_{odd}$.) Again using the properties of $\oplus$, we can re-write Equation 1 as

$$N = (x_1 p_1 \oplus x_2 p_2 \oplus \cdots \oplus x_{k_{even}} p_{k_{even}}) \oplus x_{k_{even}+1} p_{k_{even}+1} \oplus \cdots \oplus x_n p_n. \tag{5}$$

This enables us to work with the subset of all even necklaces, then introduce odd necklaces one at a time, making combinations with the already generated patterns.

When we are working only with even necklaces, we generate only even necklaces. So the bracket in Equation 5 will give us $2^{k_{even}}$ even necklaces and no odd necklaces. When we introduce the first odd necklace, we will still have $2^{k_{even}}$ even necklaces and the odd necklace combining with each one of them will give us $2^{k_{even}}$ odd necklaces. So at this stage the proportions of odd and even necklaces are half and half.

When we introduce the second odd necklace, it will combine with the existing even necklaces to form $2^{k_{even}}$ odd necklaces and with the existing odd necklaces to form $2^{k_{even}}$ odd necklaces. Remember we also have those necklaces produced before the introduction of the second odd necklace. So at this stage we have $2^{k_{even}+1}$ even necklaces and $2^{k_{even}+1}$ odd necklaces.

The process can continue and we can summarise the results as follows.

- If $k = 0$, then the original basic set and all the necklaces it can generate comprise even necklaces only.
- If $k \neq 0$, then among the necklaces in the original basic set and all the necklaces it can generate, half have odd numbers of black and white beads.

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Mentoring Scheme

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Srinivasa Ramanujan

Sheet 4

Solutions and comments

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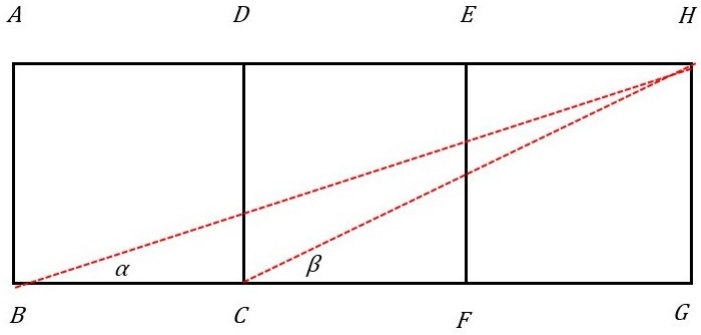
Version 1.3, Dec 2020

1. In the accompanying diagram, $ABCD$, $DCFE$ and $EFGH$ are squares. Find, with proof, the size of $\alpha + \beta$.



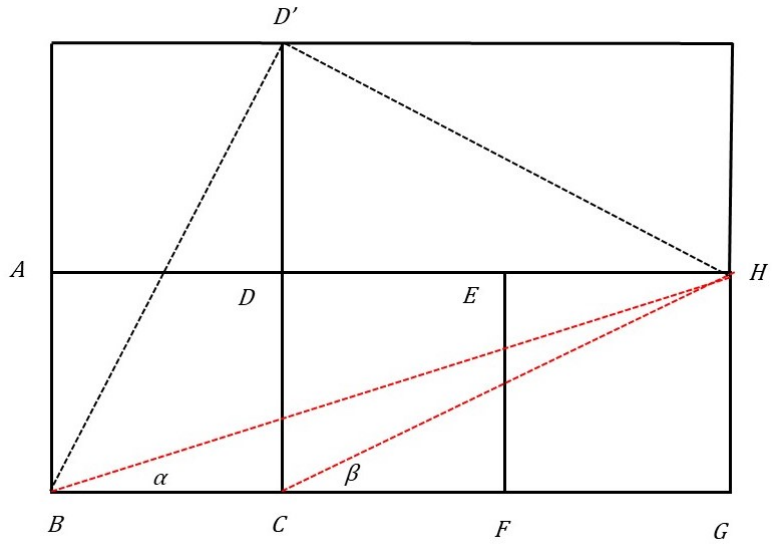
Note: With the use of an appropriate trigonometrical formula, the solution is a one-liner. However, any such formula is the result of a sophisticated argument and there are simple geometrical proofs that achieve the desired result more elegantly. It is recommended that you try to find a proof of this sort.

SOLUTION

The suggested trigonometrical solution is as follows. We have

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}} = 1.$$

It follows that $\alpha + \beta = 45°$. Moving on from here to a geometrical solution, we can investigate angles in the diagram. There is an angle that is equal to $\beta - \alpha$ but we need to construct an angle $\alpha + \beta$. Now by alternate angles, $\angle BHA = \alpha$. Hence it might be useful to construct an angle β next to it. This is the motivation for extending the configuration upwards to produce the diagram below.



In this new figure, the larger upper rectangle has the same dimensions as $ABGH$. The line segment CD is extended to meet the upper side of the upper rectangle in D'. Also the line segments BD' and $D'H$ are shown. Now, next to $\angle BHD = \angle BHA = \alpha$ we have $\angle DHD'$ and it

would be very convenient if the latter turned out to be equal to β, for then $\angle BHD'$ would be $\alpha + \beta$. The subsequent argument turns on the fact that the diagram contains three congruent right-angled triangles, namely GCH, DHD' and $CD'B$. Note that they are congruent by SAS because, in each of them, the sides including the right-angle have the following property. One is equal in length to the side of one of the squares and one has double that length.

Since corresponding angles are equal, in triangle DHD' we have $\angle DHD' = \beta$. Hence, as $\angle BHD = \alpha$, $\angle BHD' = \alpha + \beta$, as desired. Also note that $\angle HD'D = 90° - \beta$.

Turning to the third congruent triangle, we see that $\angle CD'B = \beta$ and so

$$\angle HD'B = \angle HD'D + \angle CD'B = 90°.$$

Furthermore, the hypotenuses of the congruent triangles $CD'B$ and DHD' must be equal; in other words, $D'B = HD'$. Thus triangle $HD'B$ is both isosceles and right-angled, with the immediate consequences that its two base angles are each equal to $\alpha + \beta$ and that $\alpha + \beta = 45°$.

Finally, it is worth mentioning a halfway house sort of method. If we assume that all the squares in the diagram have side 1, then by the theorem of Pythagoras, $D'B = HD' = \sqrt{5}$. Also, using triangle HBG, $HB = \sqrt{1^2 + 3^2} = \sqrt{10}$. Hence $D'B^2 + HD'^2 = HB^2$. It now follows, from the converse of the theorem of Pythagoras, that triangle HD'B is both right-angled and isosceles and the proof can be completed as before. This is less elegant than the method that uses congruence alone, as consideration of the desired result shows that it does not depend on the size of the squares.

2. Evaluate

$$\int_{\pi/6}^{\pi/3} \left(\frac{\sin x}{x}\right)^2 dx - \int_{\pi/6}^{\pi/3} \frac{\sin 2x}{x} dx.$$

SOLUTION

The integrands in this question look as though they could be troublesome to deal with. However, if they were to be combined, they would produce something reminiscient of the derivative of a quotient. In detail, we can write the original integral as

$$\int_{\pi/6}^{\pi/3} \left(\frac{\sin^2 x}{x^2} - \frac{\sin 2x}{x} dx\right) = \int_{\pi/6}^{\pi/3} \frac{-2x \sin x \cos x + \sin^2 x}{x^2} dx.$$

Now the integrand is revealed as the derivative of $\left(-\dfrac{\sin^2 x}{x}\right)$ and so the required integral is

$$\int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x^2} dx - \int_{\pi/6}^{\pi/3} \frac{\sin 2x}{x} dx = \left[-\frac{\sin^2 x}{x}\right]_{\pi/6}^{\pi/3} = \left(-\frac{3/4}{\pi/3}\right) - \left(-\frac{1/4}{\pi/6}\right) = \frac{6}{4\pi} - \frac{9}{4\pi}.$$

We conclude that the value sought is $\left(-\dfrac{3}{4\pi}\right)$.

3. Given that $p > 5$ is prime, prove that $p^4 - 1$ is divisible by 240.

Solution

In a divisibility problem like this, factorisation should always be the first port of call and here we have $p^4 - 1 = (p - 1)(p + 1)(p^2 + 1)$. The divisor needs factorising as well and this gives $240 = 2^4 \times 3 \times 5$. Since $p > 5$, each bracket is even and so we know that $p^4 - 1$ is divisible by 8. However, we know more than this: $p - 1$ and $p + 1$ are consecutive even integers and so one of them must be divisible by $4 = 2^2$. We have now guaranteed that $p^4 - 1$ has the required four factors of 2.

Next consider divisibility by 3: note that $p - 1$, p and $p + 1$ form a triple of consecutive integers, so one of them must be divisible by 3. This cannot be p, as it is a prime greater than 3, and so one of $p - 1$ and $p + 1$ is divisible by 3. Hence $p^4 - 1$ has a factor 3.

It now remains to show that $p^4 - 1$ is divisible by 5. Unfortunately, this time we cannot cherrypick one of the factors. For instance: $11 - 1 = 10 \equiv 0 \pmod 5$, $19 + 1 = 20 \equiv 0 \pmod 5$, and $7^2 + 1 = 50 \equiv 0 \pmod 5$. On the other hand, $11^2 + 1 = 122 \not\equiv 0 \pmod 5$. In other words, divisibility by 5 can pop up in any of the brackets. It is time to revisit Fermat's Little Theorem (see the Note below). This states: if q is a prime and a is a positive integer coprime to q, then $a^{q-1} \equiv 1 \pmod q$. Applying this result to the case $q = 5$ and a equal to our prime p we have $p^{5-1} = p^4 \equiv 1 \pmod 5$. This in turn implies that $p^4 - 1 \equiv 0 \pmod 5$. It just remains to note that 5 and p are coprime: this because p is a prime greater than 5. The proof is now complete but you might like to consider the excluded cases $p = 2$, $p = 3$ and $p = 5$.
Note: There are many references to Fermat's Little Theorem in books and online but ask your mentor if you have difficulty finding one. Within the mentoring programme, an early appearance was in Hanna Neumann, Sheet 5, Question 3.

4. Mrs Logistic runs a bridge club at her community centre in order to raise money for the upkeep of the Grade 1 listed building. In the current competition there are seven finalists. Mrs L needs to organise the competition so that each of the seven is paired with every other finalist in exactly two rounds. During each round three games are played and each finalist is partnered with each of the others an equal number of times. Mrs L has spent some time drawing up ad hoc lists of for each round, but without success. Investigate the problem and advise Mrs L as to how many rounds are required and how many times each player has to participate? Your solution should conclude with a table that will show Mrs L exactly how to organise the event.
Note that a game of bridge involves two pairs of players, the pairs being pitted against each other.

Solution

Let each column of the eventual table represent a three-game round. Also let each cell in a column represent one of the participants in the round. Then the number of columns is the number of rounds required. Let r be the number of times each player is going to participate and b be the number of rounds, in other words the number of columns in the table. Now $7 \times r = 4 \times b$, which will be equal to the total number of cells. Given 7 and 4 are coprime, the smallest numbers we can assign to r and b are $r = 4$ and $b = 7$.
Let us use the first seven letters in the alphabet to indicate the players. Without loss of generality we can assign a to the top position of the first 4 columns. There are 12 cells remaining in the first 4 columns. So the remaining 6 players can each appear twice, leaving 12 cells in the last three columns for them each to appear twice again. None of the the first 4 columns is identical. If

there were an identical pair, then any 3 players appearing together in two columns should not appear in the same column again. However, they each have to appear twice more and there are only 3 columns left. So, the first 4 columns cannot be identical. This means that, in each of the first 4 columns, each of the players b to g meets a twice and 4 other players once. In each case this leaves one other player not met in the first four columns. In the last 3 columns each pair that has not yet met has to appear together twice. A possible final table is shown below.

a	a	a	a	b	b	c
b	b	e	e	e	e	f
c	f	c	f	c	d	d
d	g	g	d	f	g	g

5. Imagine that a group of mentors is relaxing after the annual conference by betting on some digital horse races. (The currency is Smarties, so there is no threat to bank balances, just waistlines.) The following questions might be raised by a couple of mentors working together. A horse H enters multiple races. For each race it wins with probability p and loses with probability $q = 1 - p$. The bookies offer odds of b. This means that if we have money M before a race and bet a fraction f of that money, we end up with $M(1 + fb)$ if H wins and $M(1 - f)$ if H loses. Suppose that all races are independent.

a) Now suppose we have money M before the first race and bet a fraction f of our money on the race. Write down our expected money after the race. What value of f (as a function of b) maximises this expectation?

b) The whole meeting consists of N races. We have money M before the first race and we bet the same fraction f of our current pot of money on each race. Write down our expected money after the Nth race. What value of f (as a function of b) maximises this expectation?

c) What is the drawback with the previous strategy? In other words, what is wrong with trying to maximise the expected value of our money after many races?

d) An alternative strategy is to assume that with a large number of races, horse H will win pN of them. How much money do we end up with if we bet a fraction f of our current pot for each race and horse H wins exactly pN of them?

e) Show that maximising the quantity you found in part *d)* is equivalent to maximising the expected logarithm of our money after one race.

f) What value of f (as a function of b) maximises the expression found in part *e)*?

SOLUTION

a) The expected money is $pM(1 + fb) + qM(1 - f)$. Since $q = 1 - p$, this may be written as

$$M(p + pfb + (1 - p)(1 - f)) = M(1 + f(pb + p - 1)) = M\left(1 + pf\left(b + 1 - \frac{1}{p}\right)\right).$$

The last expression may be rewritten as $M\left(1 + pf\left(b - \left(\frac{1}{p} - 1\right)\right)\right)$. This shows that what matters is the relation between b and $\frac{1}{p} - 1$. We should bet all our money if $b \geq \frac{1}{p} - 1$ and no

money otherwise. Noting that $\frac{1}{p} - 1 = \frac{1-p}{p} = \frac{q}{p}$, we can express our recommendation as follows. We should bet all our money (implying $f = 1$) if $b \geq q/p$ and no money (implying $f = 0$) otherwise.

b) The expected money is now $M(p(1 + fb) + q(1 - f))^N$. Clearly we should optimise $p(1 + fb) + q(1 - f)$, thus we should use the same strategy as for one race. Hence we choose $f = 1$ if $b \geq q/p$ and $f = 0$ otherwise.

c) If H loses any race then we end up with no money. Even if the odds are in our favour (that is, $b \geq q/p$) we have a tiny probability of making a fortune and a large probability of losing all our money.

d) The money is $M(1 + fb)^{pN}(1 - f)^{qN}$.

e) The equivalence comes from the fact that logarithms are increasing functions. Taking the log of the answer to part *d)* we get $\ln M + pN \ln(1 + fb) + qN \ln(1 - f)$. Maximising this expression is equivalent to maximising $(p+q) \ln M + p \ln(1+fb) + q \ln(1-f) = p \ln M(1+fb) + q \ln M(1-f)$. (Note that this turns on the fact that $N > 0$.) The last expression is the expected logarithm of our money after one race.

f) Treat $p \ln(1 + fb) + q \ln(1 - f)$ as a function of f. Differentiating and equating the derivative to 0 yields

$$\frac{bp}{1 + fb} - \frac{q}{1 - f} = 0.$$

Rearranging gives $(1 - f)bp - q(1 + fb) = 0$ or $f = \frac{bp - q}{b}$. We get a maximum when $b \geq q/p$. (Ask yourself why this gives a maximum and not a minimum.) Hence we set $f = \frac{bp - q}{b}$ if $b \geq q/p$ and $f = 0$ otherwise.

6. An infection spreads through the squares of an $n \times n$ chessboard in the following manner. If a square has two or more infected neighbours, then it becomes infected itself. Note that the neighbours of a square are orthogonal to it, so that each square has at most four neighbours. Prove that, if you start with fewer than n infected squares, then the whole board cannot become infected.

SOLUTION

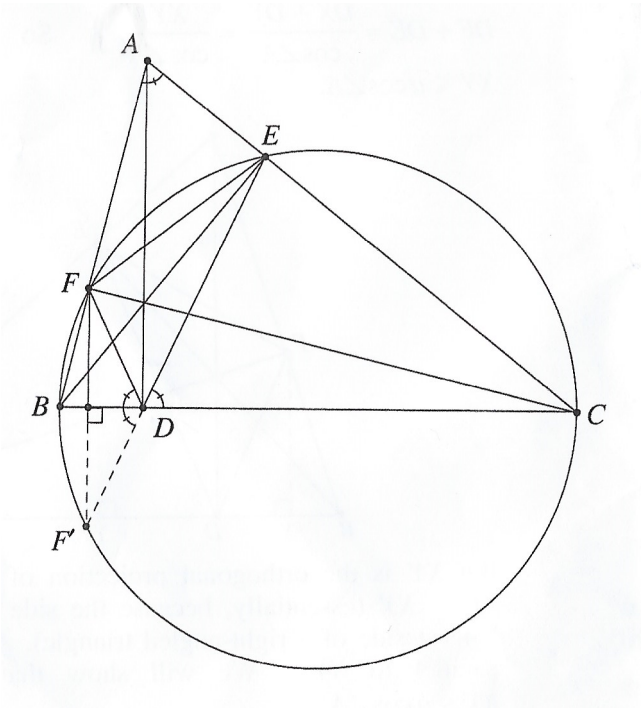
This is one of those questions where there is a host of different scenarios; far too many to investigate individually. This suggests that we look for some statement that stays true throughout the course of the infection. We cannot say anything about the total area of the infected region, as this increases in many different ways as the infection spreads. However, something else we know about each square of the board is its perimeter. Without loss of generality, assume that the side of any small square on the board is 1 unit. Consider what happens as a small square becomes infected. At least two of its sides which were on the boundary of the infected area become absorbed into its interior. Furthermore, at most two of its sides are added to the boundary of the infected area. These two facts taken together imply that this boundary cannot increase in length. If the whole board does become infected, then the perimeter of the infected region is $4n$. If you start with fewer than n infected squares, then the perimeter of the initial infected area is strictly less than $4n$ and we have just shown that this cannot increase. The result follows.

Now that the basic problem has been solved, you might like to ask yourself further questions. For instance, if you start with at least n infected squares, then will the whole board necessarily become infected? Or, is it possible to infect the whole board from a starting position where there is a row or column completely free from infected squares? Over to you to design, and answer, other questions of this sort.

7. Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leqslant BC$ and determine the triangles for which equality holds.
Note: The altitude from A is the line through A that is perpendicular to BC. The foot of this altitude is the point D where it meets BC. The other altitudes are similarly defined.
[British Mathematical Olympiad 2011 Round 1 Question 6]

SOLUTION

The diagram below illustrates the configuration, with some additions, indicated by dotted lines.



The altitudes provide a collection of (equal) right angles so there is a chance of finding a cyclic quadrilateral. In particular, $\angle AEB = \angle ADB = 90°$ and so AEDB is a cyclic quadrilateral in a circle with diameter AB. Hence $\angle EDC = \angle BAE = \angle BAC$ by exterior angle of a cyclic quadrilateral. It may similarly be proved that $\angle FDB = \angle BAC$. (Note that these three equal angles are all marked with a single arc in the diagram.)

The additions to the diagram are produced by reflecting F in the line BC to obtain F'. The triangle $FF'D$ is then completed. The reflection means that $\angle F'DB$ is equal to $\angle FDB$ and hence to all three of the angles previously noted. In particular, $\angle F'DB = \angle EDC$ and so $F'DE$ is a straight line (converse of vertically opposite angles). Now, because CF is an altitude, BFC is a right angle and, because of the reflection, the same is true of $\angle BF'C$. But BE is another altitude and so $\angle BEC = \angle BF'C = 90°$. Hence $F'BEC$ is a cyclic quadrilateral in a circle with diameter BC. We are now able to obtain an inequality, since a diameter gives the greatest distance between

two points on the circumference of a circle. We can conclude that

$$BC \geqslant F'E = F'D + DE = DF + DE.$$

This is the required result,

Equality holds if and only if $F'E$ is a diameter and this in turn is equivalent to $\angle F'CE = 90°$. Then

$$90° = \angle F'CE = \angle F'CB + \angle ECB = \angle FCB + \angle ECB = 90° - \angle ABC + \angle ACB.$$

So equality occurs if and only if angles B and C of the original triangle are equal. In terms of lengths, this is equivalent to $AB = AC$.
Remark: Note how reflection plays a useful role in this proof as it did in the purely geometrical proof in Question 1.

8. If $x \neq 2m\pi$ (m an integer), prove that

$$\sum_{k=1}^{n} \cos kx = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}.$$

State and prove a similar result for

$$\sum_{k=1}^{n} \sin kx.$$

SOLUTION

We start by noting that the denominator in the fraction on the right hand side does not involve n. Working with non-fraction terms is always easier than working with fractions. So let us aim to show that

$$\sum_{k=1}^{n} \sin \frac{1}{2}x \cos kx = \sin \frac{1}{2}nx \cos \frac{1}{2}(n+1)x.$$

The left hand side is still a sum of n terms, which do not combine together in any straightforward way. The right hand side involves n and $n+1$ only. A tempting strategy in such a situation is to seek a telescoping process, where pairs of terms cancel out. Now a product of a sine and a cosine appears in the formulae for $\sin(A+B)$ and $\sin(A-B)$. For a series to telescope we need minus signs, so consider

$$\sin(A+B) - \sin(A-B) = \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B.$$

This yields

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B), \qquad (*)$$

which can be applied to the case $A = kx$, $B = \frac{1}{2}x$. This in turn gives

$$2 \sin \frac{1}{2}x \cos kx = \sin\left(k + \frac{1}{2}\right)x - \sin\left(k - \frac{1}{2}\right)x.$$

If these terms are now added from $k = 1$ to $k = n$, then most of those on the right hand side will cancel out in pairs, that is, the series will telescope. (You should check this by writing out the first few pairs and the last few pairs on the right hand side.) We are then left with

$$\sum_{k=1}^{n} 2\sin\frac{1}{2}x\cos kx = \sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x = \sin\frac{1}{2}(2n+1)x - \sin\frac{1}{2}x.$$

Then both sides can be divided by $2\sin\frac{1}{2}x$ to obtain

$$\sum_{k=1}^{n} \cos kx = \frac{\sin\frac{1}{2}(2n+1)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}. \qquad (**)$$

However, this is not quite what the question wants: we need to turn the numerator on the right hand side back into a product of sines. To this end we can reverse the procedure in (*) above

We can follow the same sort of procedure that produced (*) above. Set $\frac{1}{2}(2n+1)x = A + B$ and $\frac{1}{2}x = A - B$. Adding and subtracting gives

$$2A = \frac{1}{2}(2n+2)x = (n+1)x \qquad \text{and} \qquad 2B = \frac{1}{2}(2n+1-1)x = nx.$$

Hence $A = \frac{1}{2}(n+1)x$ and $B = \frac{1}{2}nx$. These values can be substituted in (*) to obtain

$$2\cos\frac{1}{2}(n+1)x\sin\frac{1}{2}nx = \sin\frac{1}{2}(2n+1)x - \sin\frac{1}{2}x.$$

The final step is use this result to obtain the desired product in the numerator of (**). This gives the required result, namely

$$\sum_{k=1}^{n} \cos kx = \frac{2\sin\frac{1}{2}nx\cos\frac{1}{2}(n+1)x}{2\sin\frac{1}{2}x} = \frac{\sin\frac{1}{2}nx\cos\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x},$$

as required.

The other sum may be obtained in similar manner and it turns out that

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\frac{1}{2}nx\sin\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}.$$

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Mentoring Scheme

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Srinivasa Ramanujan

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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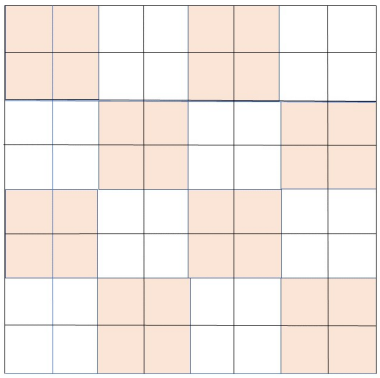
Version 1.0, Sep 2019

1. The squares of an 8×8 chessboard are mistakenly coloured in blocks of two colours as shown in the diagram below.



You need to cut the board along lines separating its rows and columns so that the standard 8×8 pattern can be reassembled from the pieces obtained. What is the minimum number of pieces into which the board needs to be cut and how should they be reassembled?

SOLUTION

The standard chessboard has no 2×1 or 1×2 rectangle coloured with the same colour. Hence every 4×4 region of the given board has to be cut both horizontally and vertically. This can be achieved by cutting down four horizontal lines and four vertical lines. When this is done, it produces four 1×1 squares, twelve 1×2 rectangles and nine 2×2 squares, each coloured in a way that could be fitted into a standard colouring. This provides 25 pieces altogether and is a minimum because the eight cuts described above *have* to be made. It remains to verify that this minimum can be achieved. In other words, can the 25 pieces be reassembled into a chessboard coloured in the usual way?

This final stage is left as an exercise for the reader. You can either give a careful description of the reassembly or you can make your own jigsaw and fit it together.

2. The function $x \mapsto f(x)$ is a polynomial function over the real numbers. Find an expression for $f(x)$ if

$$|f(x)|^2 = x^4 - 8x^3 + 22x^2 - 24x + 9.$$

SOLUTION

First consider $|f(x)|$ and recall that it is equal to $f(x)$ or $-f(x)$, according as to whether $f(x)$ is non-negative or negative. Thus, when the modulus is squared, we obtain $(f(x))^2$ in both cases. Since f is polynomial and produces a quartic (degree 4) polynomial when squared, it must be a quadratic. Furthermore f must have leading coefficient 1 (why?). It is now straightforward to obtain the quadratic explicitly and we might as well do so as efficiently as possible.

Note that $|f(1)|^2 = 1 - 8 + 22 - 24 + 9 = 0$, so $f(1) = 0$, and hence $x - 1$ is a factor of $f(x)$. Furthermore

$$(f(x))^2 = (x-1)(x^3 - 7x^2 + 15x - 9) = (x-1)(x-1)(x^2 - 6x + 9) = (x-1)^2(x-3)^2.$$

Thus $f(x) = \pm(x-1)(x-3) = \pm(x^2 - 4x + 3)$.
Note: It is a very useful skill to be able to do such factorisations mentally.

3. Fibonacci sequences are defined by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \quad \text{where } n \geqslant 1.$$

Consider the particular Fibonacci sequence where $u_1 = u_2 = 1$. Show that, for any positive integer k, there is a term of this sequence that is divisible by k.

Solution

On starting this question, we do not know how far along the sequence we are going to have to go. One way of reducing the number of items we have to deal with is to work modulo k. (You may have already spotted this as a useful strategy by exploring terms at the start of the sequence.) Consider the (infinite) list of all possible pairs (u_n, u_{n+1}). When this list is reduced modulo k, there are at most k^2 distinct pairs (a, b), where each of a and b is a member of the set $\{0, 1, \ldots, (k-1)\}$.

In what follows the remainder of u_n on division by k is denoted by $\overline{u}_n$. Consider the sequence of pairs of such remainders:

$$(\overline{u}_1, \overline{u}_2), \quad (\overline{u}_2, \overline{u}_3), \quad (\overline{u}_3, \overline{u}_4), \quad \ldots, \quad (\overline{u}_n, \overline{u}_{n+1}), \quad \ldots \tag{1}$$

By the pigeonhole principle, and using the observation in the first paragraph, there must be be a repeated pair in this sequence. Let the first such pair be $(\overline{u}_m, \overline{u}_{m+1})$. We aim to show that this pair is actually the first pair in (1), namely $(1, 1)$. In order to narrow down the number of pairs we need to think about, we argue by contradiction and initially assume that $m > 1$.

As we consider how to proceed, it is worth observing that no use has yet been made of the Fibonacci recurrence relation. This can provide links between the pairs in the sequence (1). By the definition of $(\overline{u}_m, \overline{u}_{m+1})$, there is a pair $(\overline{u}_l, \overline{u}_{l+1})$ with $l > m$ and

$$(\overline{u}_l, \overline{u}_{l+1}) = (\overline{u}_m, \overline{u}_{m+1}). \tag{2}$$

Precisely, this means that $\overline{u}_l = \overline{u}_m$ and $\overline{u}_{l+1} = \overline{u}_{m+1}$. Now the Fibonacci relation tells us that $u_{l-1} = u_{l+1} - u_l$ and $u_{m-1} = u_{m+1} - u_m$, so it follows from (2) that the remainders of u_{l-1} and u_{m-1} on division by k are equal. In other words $\overline{u}_{l-1} = \overline{u}_{m-1}$ and consequently

$$(\overline{u}_{l-1}, \overline{u}_l) = (\overline{u}_{m-1}, \overline{u}_m).$$

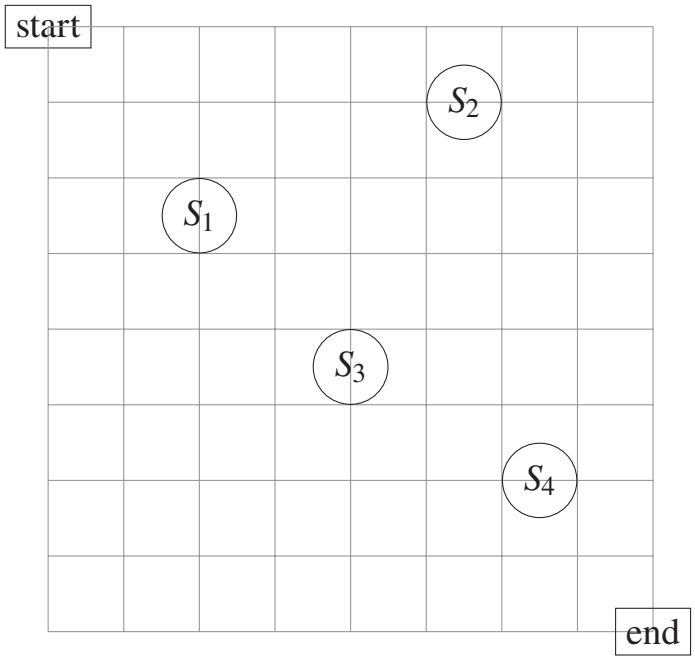
However, the pair $(\overline{u}_{m-1}, \overline{u}_m)$ comes earlier in the sequence (1) than the pair $(\overline{u}_m, \overline{u}_{m+1})$ and so $(\overline{u}_m, \overline{u}_{m+1})$ is not the first repeated pair. ***Contradiction***.

We conclude that it was wrong to assume that $m > 1$ and so the first repeated pair in the sequence (1) is $(1, 1)$. Suppose that the repeat is $(\overline{u}_t, \overline{u}_{t+1})$, where $t > 1$, that is $(\overline{u}_t, \overline{u}_{t+1}) = (1, 1)$. This means that both u_t and u_{t+1} leave remainder 1 on division by k. It follows immediately that their difference is divisible by k. But $u_{t+1} - u_t = u_{t-1}$ and so u_{t-1} is divisible by k and is the term that we are seeking.

Historical note: You may have met Leonardo of Pisa in Question 4 of Srinivasa Ramanujan, Sheet 3. There is a huge literature treating Fibonacci sequences, kick-started by his charming, or possibly disconcerting, problem about rabbit breeding. More importantly, Leonardo played a decisive role in the development of Western culture, thanks to his book Liber Abaci. *This was*

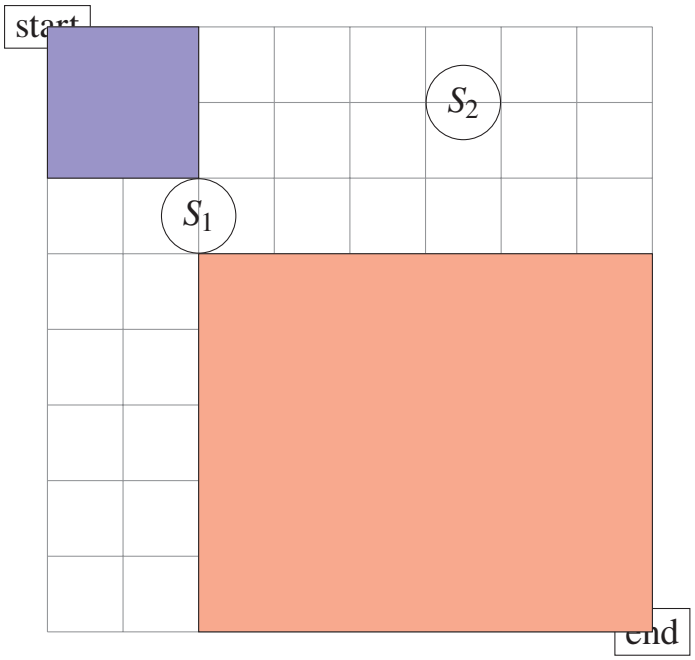
completed in 1202 CE and introduced Indo-Arabic numerals into Europe. Interestingly, Leonardo was not primarily a scholar but a businessman, apparently working in his father's import-export firm, which was based in North Africa. It is ironic that, as shown by recent research, Leonardo's innovative approach to arithmetic was slow to catch on in European commerce: it was thought to be less secure than established practice. Records could be altered with comparative ease, especially as the forms of numerals were not standardised immediately.

4. Johnny went to a playground and came across the activity illustrated below. Each line segment which forms the 8×8 grid represents a walkway. The aim of the game is to walk from the start to the end without crossing any section where a (plastic) snake (S_1 to S_4) is marked. Johnny is allowed to proceed only towards the right-hand side or the bottom of the grid. How many distinct routes could Johnny take?

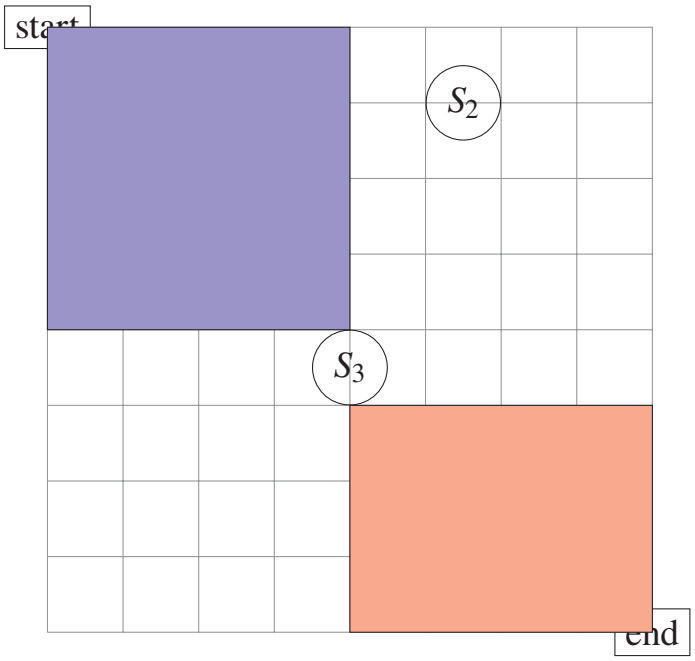


SOLUTION

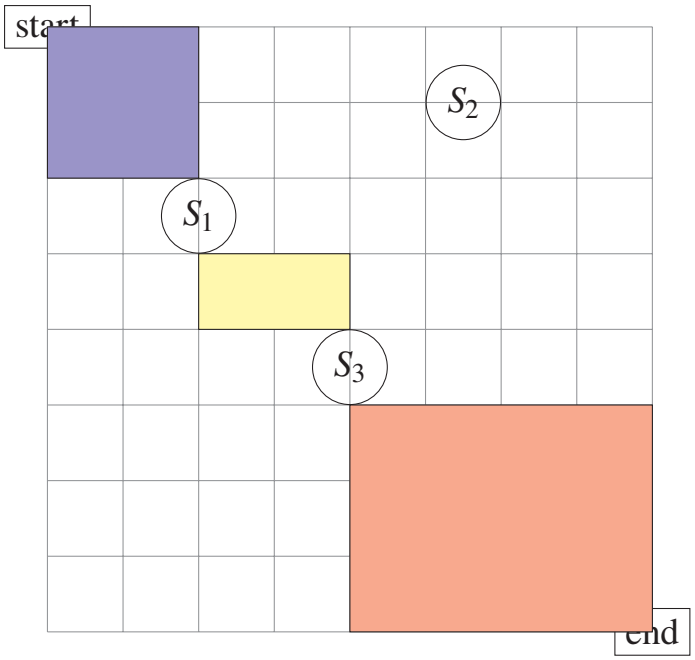
First observe that routes to S_2 do not involve any segments of those passing S_1 and S_3. Routes to S_4 include segments of routes passing S_1, S_2 and S_3.



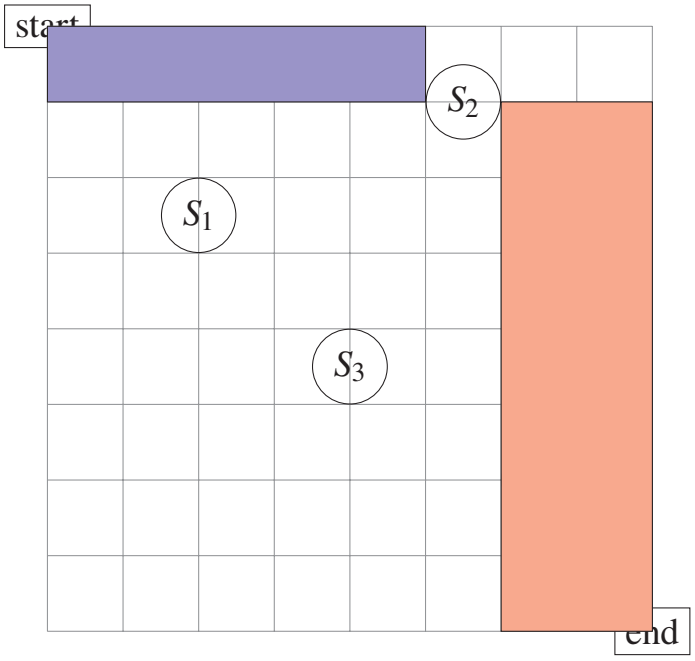
The blue box represents the routes which arrive at S_1 and the ginger box represents the routes subsequently prevented by S_1. Similar colour conventions hold in the diagrams that follow. We observe that there are $\binom{4}{2} \times \binom{11}{5}$ routes prevented by S_1.



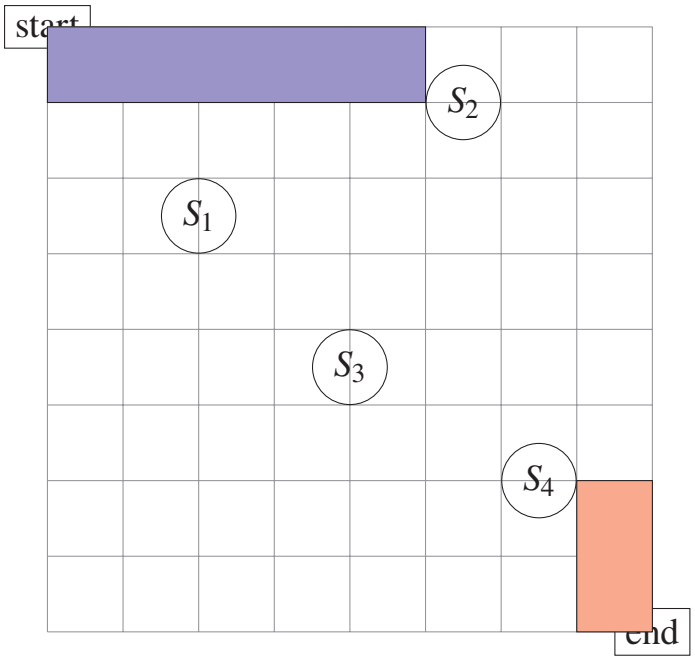
Similarly we see that $\binom{8}{4} \times \binom{7}{3}$ routes are prevented by S_3.



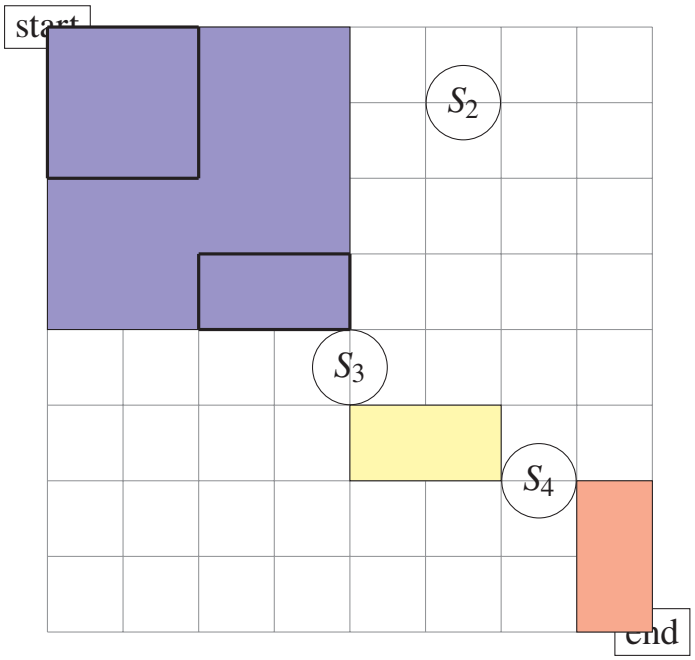
However, this next diagram shows that, of these, there are $\binom{4}{2} \times \binom{3}{1} \times \binom{7}{3}$ routes already eliminated by S_1.



There are $\binom{6}{1} \times \binom{9}{7}$ routes prevented by S_2.



There would have been $\binom{12}{6} \times \binom{3}{2}$ routes prevented by S_4. However, $\binom{6}{1} \times \binom{3}{2}$ of these are already prevented by S_2 and $\binom{4}{2} \times \binom{7}{3} \times \binom{3}{2}$ of these are already prevented by S_1.
All that remains is to find those routes arriving at S_4, that are prevented by S_3 but not S_1.



The number of routes arriving at S_4 prevented by S_3 but not S_1 is $(\binom{8}{4} - \binom{4}{2} \times \binom{3}{1}) \times \binom{3}{1} \times \binom{3}{2}$

Altogether, the number of possible routes is

$$\binom{16}{8} - \binom{4}{2} \times \binom{11}{5} - \binom{6}{1}\binom{9}{7} - \left[\binom{8}{4} - \binom{4}{2}\binom{3}{1}\right]\binom{7}{3} - \left[\binom{12}{6} - \binom{4}{2}\binom{7}{3} - \binom{6}{1} - \left(\binom{8}{4}\binom{3}{1} - \binom{4}{2}\binom{3}{1}\binom{3}{1}\right)\right]\binom{3}{2}$$

$$= 6406.$$

5. *The angle bisector theorem is a useful result that is stated here for those readers who have not met it before.* Suppose that, in the triangle ABC, the internal bisector of the angle at C meets the side AB at D. Then $\frac{AD}{DB} = \frac{AC}{CB}$. A similar result holds for the external bisector of the angle at C.

a) Points A and B are fixed and P is a point, not lying on AB, such that

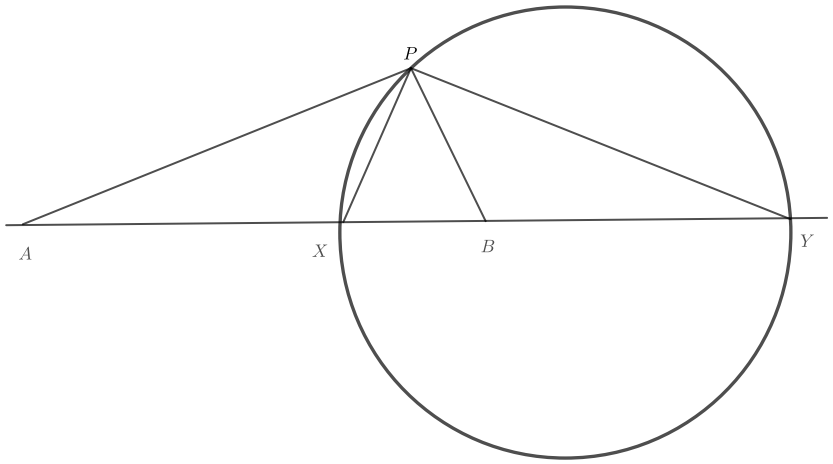
$$\frac{AP}{PB} = k, \quad \text{where } k \text{ is a positive constant with } k \neq 1. \tag{*}$$

The internal and external bisectors of $\angle APB$ meet AB in points X and Y respectively. Show that such a point P lies on a (fixed) circle on XY as diameter. (Such a circle is often called a circle of Apollonius.)

b) Show that the locus of a point satisfying the condition in part *a)* is the whole of the circle obtained in that previous result.

SOLUTION

a) The diagram below illustrates the required configuration.



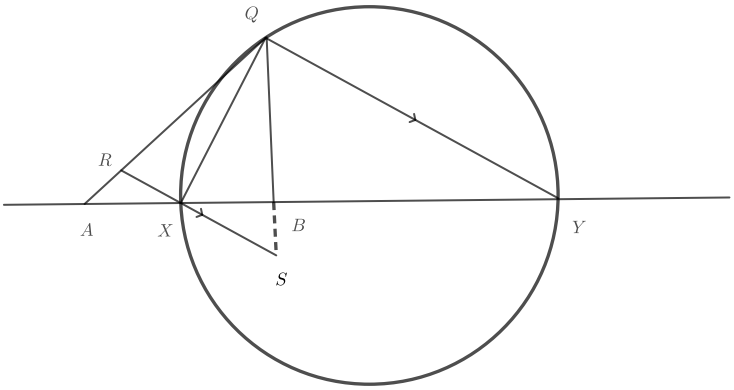
The angle bisector theorem ensures that

$$\frac{AX}{XB} = \frac{AY}{YB} = \frac{AP}{PB} = k. \tag{1}$$

Hence the points X and Y are fixed; we also know that $\angle XPY = 90°$. Thus, as required, we know that the point P lies on a fixed circle having XY as diameter.

b) At first sight there seems nothing more to do. However, what we are being asked here is to show that the set of points satisfying condition (*) is precisely the set of points comprising the circumference of the circle. The missing part of the proof is to start with any point Q, say, on the circle and then show that $\frac{AQ}{QB} = k$. The following diagram shows such a point Q: to the previous diagram is added the line through X parallel to QY. This meets QA and QB (produced) in the points R and S respectively. This may seem like an arbitrary choice but

the idea is that parallels help to set up useful similar triangles. Further, the first part of proof suggests that we should aim to prove that QX is the internal bisector of $\angle AQB$.



First observe that that triangles AXR and AYQ are similar. Hence

$$\frac{AX}{AY} = \frac{XR}{YQ} = \frac{RA}{QA}. \tag{2}$$

Again, triangles BXS and BYQ are similar so

$$\frac{BX}{BY} = \frac{XS}{YQ} = \frac{SB}{QB}. \tag{3}$$

Dividing appropriately,

$$\frac{AX}{AY}\Big/\frac{BX}{BY} = \frac{XR}{YQ}\Big/\frac{XS}{YQ} \tag{4}$$

Cancelling shows that the right-hand side of (4) is just XR/XS; further, (1) implies that the left hand side of (4) is $k/k = 1$. Hence $XR = XS$. Further note that, since XY is a diameter, $\angle XQY = 90°$. The parallel lines then ensure that $\angle QXS = 90°$. Combining these two results, we see that triangles QXS and QXR are congruent (SAS). Therefore the angles at Q are equal and consequently QX is the internal bisector of $\angle AQB$. Applying the angle bisector theorem again, we have

$$\frac{AX}{XB} = \frac{AQ}{QB}.$$

It follows from (1) that $\dfrac{AQ}{QB} = k$, as required.

6. Find all solutions of the following pair of simultaneous equations, where x and y are real.

$$(x+y)^{\frac{2}{3}} + 2(x-y)^{\frac{2}{3}} = 3(x^2-y^2)^{\frac{1}{3}} \tag{1}$$

$$3x - 2y = 13 \tag{2}$$

SOLUTION

It is tempting to use equation (2) to express one variable in terms of the other. However, it turns out that this does not help us very much. What about equation (1)? First note that $x^2 - y^2 = (x - y)(x + y)$. This suggests that a start might be made by dividing through by $(x^2 - y^2)^{\frac{1}{3}} = (x - y)^{\frac{1}{3}}(x + y)^{\frac{1}{3}}$. The resulting equation would then include a couple of similar terms. However, the mention of dividing should dictate caution. Could $(x^2 - y^2)^{\frac{1}{3}}$ be zero? Could the equations be simultaneously be satisfied if $x = y$? What if $x = -y$? Check carefully that this cannot happen.

Since it is safe to divide, we do so and obtain

$$\frac{(x + y)^{\frac{1}{3}}}{(x - y)^{\frac{1}{3}}} + 2\frac{(x - y)^{\frac{1}{3}}}{(x + y)^{\frac{1}{3}}} = 3. \tag{3}$$

Now set

$$X = \frac{(x + y)^{\frac{1}{3}}}{(x - y)^{\frac{1}{3}}} = \left(\frac{x + y}{x - y}\right)^{\frac{1}{3}}.$$

Recall that $X \neq 0$ because $x \neq -y$. In terms of X, (3) becomes the valid equation

$$X + \frac{2}{X} = 3. \tag{4}$$

Multiplying up yields the quadratic equation

$$X^2 - 3X + 2 = 0, \tag{5}$$

with roots $X = 1$ and $X = 2$. Consider first the case $X = 2$. This is equivalent to

$$\left(\frac{x + y}{x - y}\right)^{\frac{1}{3}} = 2 \quad \text{or} \quad \left(\frac{x + y}{x - y}\right) = 8, \quad \text{implying} \quad 7x - 9y = 0. \tag{6}$$

Combining this with (2), we have $3x - 2 \times \frac{7}{9}x = 13$. On multiplying up this equation becomes $(27 - 14)x = 9 \times 13$. Thus $x = 9$ and it follows that $y = 7$. After this rather tortuous argument, it would be prudent to check these solutions. Clearly the values $x = 9$ and $y = 7$ satisfy equation (2). Then, substituting in the left-hand side of (1) gives

$$(9 + 7)^{\frac{2}{3}} - 2 \times (9 - 7)^{\frac{2}{3}} = 16^{\frac{2}{3}} + 2 \times 2^{\frac{2}{3}} = (2^2 + 2)2^{\frac{2}{3}} = 3 \times 2 \times 2^{\frac{2}{3}} = 3 \times 2^{\frac{5}{3}}.$$

Again, substituting in the right-hand side of (1) gives

$$3(81 - 49)^{\frac{1}{3}} = 3 \times 32^{\frac{1}{3}} = 3 \times (2^5)^{\frac{1}{3}} = 3 \times 2^{\frac{5}{3}}.$$

Thus the presumed solutions do actually satisfy both equations.

The case $X = 1$ leads to the solutions $x = \frac{13}{3}$, $y = 0$. Verify that these values satisfy both equations: the checks needed are simpler in this case.

7. The set of real numbers is split into two subsets which do not intersect. Prove that for each pair (m, n) of positive integers, there are real numbers $x < y < z$ all in the same subset such that $m(z - y) = n(y - x)$.

[British Mathematical Olympiad 2012 Round 2 Question 3]

Solution

First we will prove it for the case $m = n$. So we have to show that one of the subsets contains an arithmetic progression of length 3. Call the subsets X and Y. Suppose $a < b$ are in the same subset, say they are in X. Then, if there is no arithmetic progression of length 3, none of $2a - b$, $\frac{1}{2}(a + b)$ or $2b - a$ can be in X. But then $2a - b$, $\frac{1}{2}(a + b)$, $2b - a$ is a three-term arithmetic progression in Y.

Now we deal with the case where $m \neq n$. We have shown that one of the subsets, say X, contains a three-term arithmetic progression. We can write this arithmetic progression as cm, $cm + dm$, $m + 2dm$. Suppose there are are no $x < y < z$ in X satisfying $m(z - y) = n(y - x)$ - call this equation R.

Consider $cm + dm + dn$. Then $x = cm$, $y = cm + dm$, $z = cm + dm + dn$ satisfy equation R and x and y are in X, so $z = cm + dm + dn$ is in Y.

Now consider $cm + 2dm + dn$. Then $x = cm + dm$, $y = cm + 2dm$, $z = cm + 2dm + dn$ satisfy equation R and x and y are in X, so $z = cm + 2dm + dn$ is in Y.

Finally consider $cm + 2dm + 2dn$. Then $x = cm$, $y = cm + 2dm$, $z = cm + 2dm + 2dn$ satisfy equation R and x and y are in X, so $z = cm + 2dm + 2dn$ is in Y.

But now $x = cm + dm + dn$, $y = cm + 2dm + dn$, $z = cm + 2dm + 2dn$ are in Y and satisfy equation R. We have shown that either X or Y contains a triple (x, y, z) satisfying equation R, as required.

8. *This question offers the opportunity to investigate a pair of somewhat unusual graphs. Some aspects may require a more advanced analysis that goes beyond your current knowledge. But you are encouraged to see how far you can get.*

On the same axes sketch the graphs of the following functions.

$$f : x \mapsto x^{|\ln x|} \qquad \text{and} \qquad g : x \mapsto x|\ln x|$$

The domain of each function should be as large as possible. Mark all special features of each graph, showing your reasoning in each case.

Solution

Because this is an exploratory question, some details will be omitted from the solution. Before you start, it will be useful to have a clear picture of the graphs $y = x$ and $y = \ln x$ and their relation to each other.

For clarity, set

$$y_1 = x^{|\ln x|} \qquad \text{and} \qquad y_2 = x|\ln x|. \tag{1}$$

Because of the logarithms, each function will have domain $\mathbb{R}^+$, the set of positive real numbers. Again, the modulus signs need to be removed so that we can do the necessary algebra. In the case of y_1, we have

$$y_1 = x^{-\ln x} \text{ if } 0 < x < 1 \qquad \text{and} \qquad y_1 = x^{\ln x} \text{ if } x \geqslant 1. \tag{2}$$

Since $x > 0$ and it is always raised to an non-negative power, $y_1 > 0$. We shall go on to consider what happens as $x \to 0^+$ but first we consider the derivative of y_1. *(For the 0^+ notation see the Note below.)* The usual way to deal with a variable power is to take the logarithm of the function. We thus have

$$\ln y_1 = -\ln x(\ln x) = -(\ln x)^2 \text{ if } 0 < x < 1 \text{ and } \ln y_1 = \ln x(\ln x) = (\ln x)^2 \text{ if } x \geqslant 1. \tag{3}$$

Differentiating implicitly and then tidying up gives

$$\frac{dy_1}{dx} = -\frac{2x^{-\ln x}\ln x}{x} \text{ if } 0 < x < 1 \qquad \text{and} \qquad \frac{dy_1}{dx} = \frac{2x^{\ln x}\ln x}{x} \text{ if } x \geqslant 1. \tag{4}$$

First note that, if $0 < x < 1$, then $\dfrac{dy_1}{dx} > 0$ because $\ln x < 0$. Also, if $x \geqslant 1$, then $\dfrac{dy_1}{dx} \geqslant 0$ because $\ln x \geqslant 0$. It follows that the function is everywhere increasing but with a stationary point when $x = 1$ and $\ln x = 0$. We further note that this stationary point must be a point of inflection.

Now examine how y_1 behaves near the origin. As $x \to 0^+$, $\ln x \to -\infty$ and $-\ln x \to \infty$. It follows from (2) that $y_1 \to 0$. Turning to large values of x, we also see from (2) that $y_1 \to \infty$ as $x \to \infty$. As a final step, use (4) to examine *how* the graph approaches the origin. As $x \to 0^+$, $-2x^{-\ln x - 1} \to 0$ and $\ln x \to -\infty$. Which "wins"? It can be shown (though you are not expected to do so here) that powers always "beat" logarithms & so $\dfrac{dy_1}{dx} \to 0$.

We can now consider the other function. This time we have

$$y_2 = -x\ln x \text{ if } 0 < x < 1 \qquad \text{and} \qquad y_2 = x\ln x \text{ if } x \geqslant 1. \tag{5}$$

The signs of the logarithms imply that $y_2 \geqslant 0$ with $y_2 = 0$ when $x = 1$. This must be a local minimum of the function. Next consider the derivative: in this case

$$\frac{dy_2}{dx} = -\ln x - 1 \text{ if } 0 < x < 1 \qquad \text{and} \qquad \frac{dy_2}{dx} = \ln x + 1 \text{ if } x > 1. \tag{6}$$

Note first that if $x > 1$, then $\dfrac{dy_2}{dx} > 0$ and the function is increasing for these values of x. Next observe that $\dfrac{dy_2}{dx} = 0$ when $\ln x = -1$ or equivalently $x = 1/\mathrm{e}$. Furthermore,

$$\frac{dy_2}{dx} > 0 \text{ if } 0 < x < 1/\mathrm{e} \qquad \text{and} \qquad \frac{dy_2}{dx} < 0 \text{ if } 1/\mathrm{e} < x < 1. \tag{7}$$

It follows that the stationary point at $x = 1/\mathrm{e}$ is a local maximum. Consider again what is happening near $x = 1$. As $x \to 1^-$, $\dfrac{dy_2}{dx} \to -1$ but, as $x \to 1^+$, $\dfrac{dy_2}{dx} \to 1$. (*See the Note again.*) Thus at $x = 1$ the graph has a sharp change of direction; in other words the function is not differentiable there. That is why another type of argument was needed to confirm the existence of a local minimum at $x = 1$. Now we can ask: how does the graph behave near the origin?

As x decreases from $1/\mathrm{e}$ and approaches 0, the sign of the derivative implies that the values of y_2 go on decreasing. So it is likely that $y_2 \to 0$ as $x \to 0$ from above. However, that is not quite enough for us to know precisely what the limit is. As above, the argument depends on the limiting behaviour of powers and logarithms and it can be shown that the graph behaves as suspected. In addition, $\dfrac{dy_2}{dx} \to \infty$ as $x \to 0^+$ and so, as the graph approaches the origin, the gradient becomes closer and closer to the vertical (see (6)).

Finally we consider how the two graphs relate to each other; in particular, where do they intersect? Now, because ln is a one-one function, $y_1 = y_2$ precisely when $\ln y_1 = \ln y_2$. Using the latter,

$$\ln(x^{|\ln x|}) = \ln(x|\ln x|) \qquad \text{precisely when} \qquad |\ln x| \ln x = \ln x + \ln|\ln x|. \tag{8}$$

Note that, since the graph of y_1 "humps up" between 0 and 1, while the values of y_2 increase, it is likely that the curves intersect in this range. For such values (8) becomes

$$-(\ln x)^2 = \ln x - \ln(-\ln x). \tag{9}$$

It is hard to see how to handle the second term on the right-hand side: what if it were equal to 0? With this assumption, we have $-\ln x = -1$ (because $\ln x \neq 0$). Hence at the point of intersection, $x = 1/\mathrm{e}$. Substituting this value in (9) shows that this is indeed a solution of the equation; the coordinates of the point of intersection are $(1/\mathrm{e}, 1/\mathrm{e})$. Since y_1 is increasing and y_2 has a local maximum at $x = 1/\mathrm{e}$, the graphs cannot intersect again when $x < 1$. Is there another intersection?

If $x \geqslant 1$, then (8) becomes $(\ln x)^2 = \ln x + \ln(\ln x))$. This time, taking $x = \mathrm{e}$ makes both sides of the equation equal to 1 and so there is another point of intersection at (e,e). Furthermore, (4) and (6) show that $x = \mathrm{e}$ gives $\dfrac{dy_1}{dx} = \dfrac{dy_2}{dx} = 2$. Hence, not only do the graphs intersect when $x = \mathrm{e}$, they actually touch.

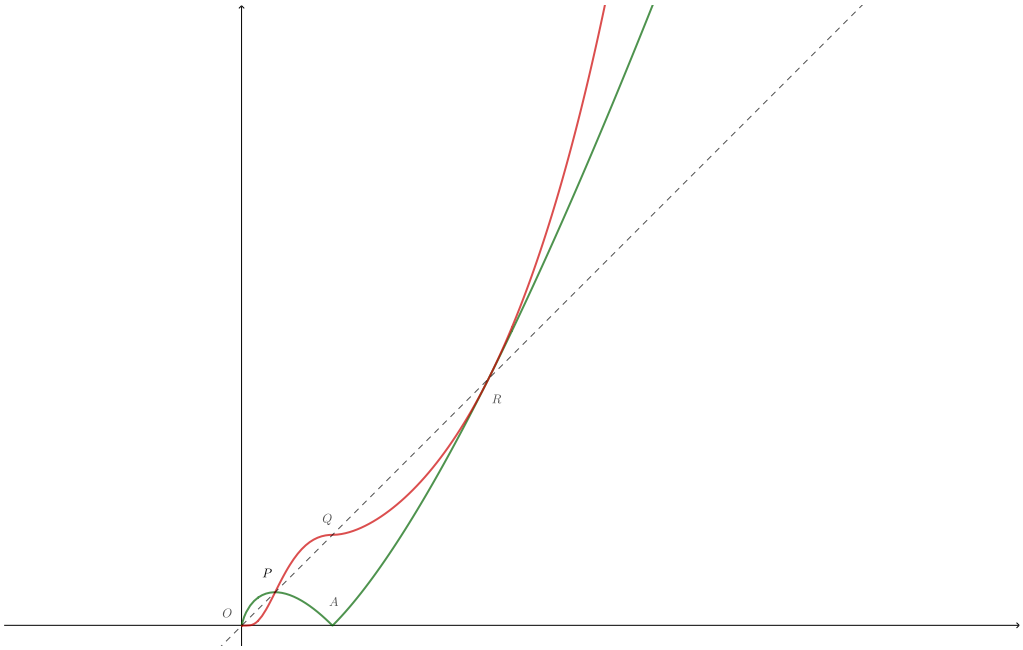
So far we have essentially spotted solutions: could there be other intersections? Since y_1 has $|\ln x|$ as an exponent and y_2 has $|\ln x|$ as a factor, it is likely that the values of y_1 grow more rapidly than those of y_2 and consequently the graphs diverge when $x > \mathrm{e}$. The graphs are shown below, after the comments.

Note: The notation $x \to a^+$ means that we consider what happens as x tends to a from the right, that is through values of x that are larger than a. In the case of $x \to a^-$, we have x approaching a from the left.

Remark: In an exploration of this type, it is not really cheating to use graph-sketching software or a graphical calculator to confirm your hunches about the curves. The challenge then becomes to ***explain*** *what you observe. Further, those sophisticated aids are not essential: testing well-chosen values of x with a scientific calculator can add to your understanding, even though it does not constitute proof.*

In the diagram the graph of y_1 is shown in red and that of y_2 in green. The dotted line is $y = x$, inserted to clarify the relationship between the curves. The marked points are:

$$O = (0,0) \quad A = (1,0) \quad P = (1/\mathrm{e}, 1/\mathrm{e}) \quad Q = (1,1) \quad R = (\mathrm{e}, \mathrm{e}).$$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Srinivasa Ramanujan

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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Version 2.1, Feb 2021

1. *A puzzle devised by Lewis Carroll*
The Dodo says that the Hatter tells lies.
The Hatter says that the March Hare tells lies.
The March Hare says that both the Dodo and the Hatter tell lies.
Who is telling the truth?

SOLUTION

A simple procedure is to assume in turn that each character is telling the truth. Starting with the Dodo: if he is telling the truth, then the Hatter is lying. But the (lying) Hatter says that the March Hare tells lies. Hence the March Hare tells the truth. But he says that the Dodo lies, contradicting our original assumption.

Next assume that the Hatter tells the truth. It follows that the March Hare tells lies. Now consider his statement. If he is lying, then at least one of the Dodo and the Hatter tells the truth. No contradiction here, as we are assuming that the Hatter tells the truth.

Finally assume that the March Hare is telling the truth. Hence both the Dodo and the Hatter are lying. Thus the Dodo's statement is untrue and the Hatter tells the truth. But then he (truthfully) says that the March Hare lies, which contradicts the current assumption.

Hence it is the Hatter who is telling the truth.

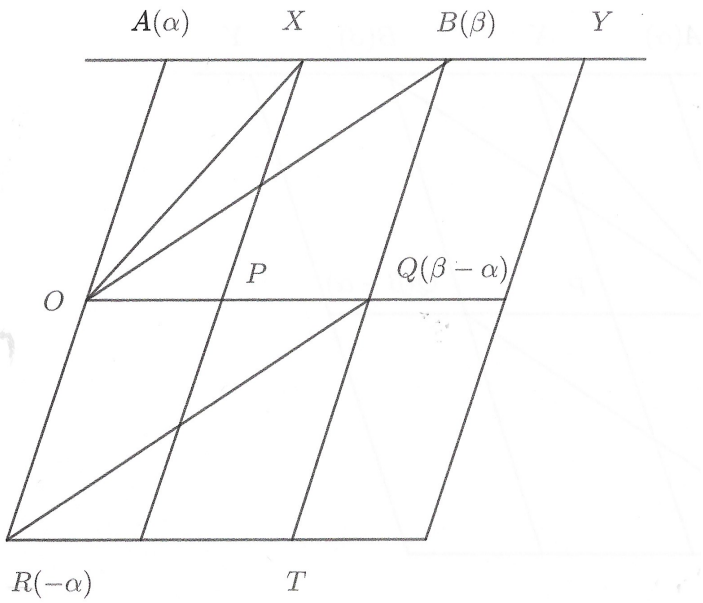
2. If you did Question 5 on Ramanujan Sheet 5, then you met the idea of a circle of Apollonius. Such circles have have particularly nice equations in the Argand diagram.
Reminder: Suppose A and B are fixed points. A circle of Apollonius is the locus of a point P such that $AP/PB = k$, where $k \neq 1$ is a positive constant. The circle intersects the line through A and B. Furthermore, the diameter of the circle lies along this line. If necessary, check these details by referring to the solutions for Sheet 5.

a) Suppose that the points A and B in the Argand diagram represent the complex numbers α and β respectively. If the variable point Z represents the complex number z, then write down the equation of the circle in terms of z, α and β.
b) Find the radius and centre of the circle in terms of α, β and k.

SOLUTION

a) The equation of the circle is $|z - \alpha| = k|z - \beta|$.

b) The diagram below represents part of the Argand diagram. The Apollonius circle is not shown but the points X and Y are the points where it intersects the line through A and B. Since XY is the diameter of the circle (see the Reminder above), the centre of the circle is the midpoint of XY and the radius is $\frac{1}{2}XY$.



In the diagram O is the origin and represents 0. Also note that lines that appear to be parallel in the diagram are in fact parallel.

The segment AO is extended to R so that $AO = OR$. Hence R represents the number $-\alpha$.

The quadrilaterals $OQBA$ and $ORTQ$ are congruent parallelograms. Hence $ORQB$ is a parallelogram with diagonal OQ and so the point Q represents the number $\beta - \alpha$.

Since X lies on the circle, it divides the segment AB in the ratio $k : 1$. It follows that $AX = \dfrac{k}{k+1}AB$ and $XB = \dfrac{1}{k+1}AB$. However, we have noted that $OQBA$ is a parallelogram implying that $OP = \dfrac{k}{k+1}OQ$. Consequently, P represents the complex number $\dfrac{k}{k+1}(\beta-\alpha)$.

Further, $OPXA$ is another parallelogram, implying that X represents $\alpha + \dfrac{k}{k+1}(\beta-\alpha)$.

By a similar argument it can be shown that Y represents $\alpha + \dfrac{k}{k-1}(\beta-\alpha)$.

Combining the results for X and Y shows that the number represented by the centre of the circle is

$$\frac{1}{2}\left[2\alpha + (\beta-\alpha)\left(\frac{k}{k+1}+\frac{k}{k-1}\right)\right] = \frac{k^2\beta-\alpha}{k^2-1}.$$

The results above also show that the radius of the circle is given by

$$\frac{1}{2}\left|\alpha + \frac{k}{k-1}(\beta-\alpha) - \alpha - \frac{k}{k+1}(\beta-\alpha)\right| = k\left|\frac{\beta-\alpha}{k^2-1}\right|.$$

3. *a)* Each of Isabelle and Annabelle thinks of three positive integers. For each pair of her numbers, Isabelle writes down the greatest common divisor of the two numbers. For each pair of her numbers, Annabelle writes down the least common multiple of the two numbers. If both Isabelle and Annabelle write down the same three numbers, then prove that these three numbers are all equal to each other.

b) Can the analogous result be proved if, initially, each of Isabelle and Annabelle thinks of four positive integers instead?

SOLUTION

a) In this question there could be confusingly many equal integers. This suggests that it might be profitable to argue by argue by contradiction and to assume that the numbers written down are not all equal to each other. Next we define some notation.

Suppose that Isabelle's positive integers are s_1, s_2 and s_3 and Annabelle's are n_1, n_2 and n_3. Denote the three numbers written down by both Isabelle and Annabelle by w_1, w_2 and w_3. Use the obvious notation for the greatest common divisors and the least common multiples. For example, the greatest common divisor of s_1 and s_2 is denoted by $\gcd(s_1, s_2)$ and the least common multiple of n_1 and n_2 is denoted by $\operatorname{lcm}(n_1, n_2)$.

We are assuming that w_1, w_2 and w_3 are not all equal to the same number. This means that $\gcd(s_1, s_2)$, $\gcd(s_1, s_3)$ and $\gcd(s_2, s_3)$ are not identical. Therefore there must be a prime p such that the highest powers of p which divide each of these three numbers are not identical. To make the situation more precise, let the highest powers of p which divide s_1, s_2 and s_3 be a, b and c respectively. In other words,

$$s_1 = p^a \times E_1 \qquad s_2 = p^b \times E_2 \qquad s_3 = p^c \times E_3,$$

where each E_i denotes the rest of the prime factorisation of the corresponding s_i. Again without loss of generality, we can assume that $a \leqslant b \leqslant c$. Then

$$\gcd(s_1, s_2) = p^a \times F_1 \qquad \gcd(s_1, s_3) = p^a \times F_2 \qquad \gcd(s_2, s_3) = p^b \times F_3. \tag{1}$$

Here the F_i (and the G_i and H_i subsequently) play a role analogous to that of the E_i; they denote the rest of the corresponding factorisation. Also note that we must have $a < b$. (Make sure that you understand why this must be so.)

Now consider Annabelle's numbers and suppose similarly that

$$n_1 = p^r \times G_1 \qquad s_2 = p^s \times G_2 \qquad s_3 = p^t \times G_3,$$

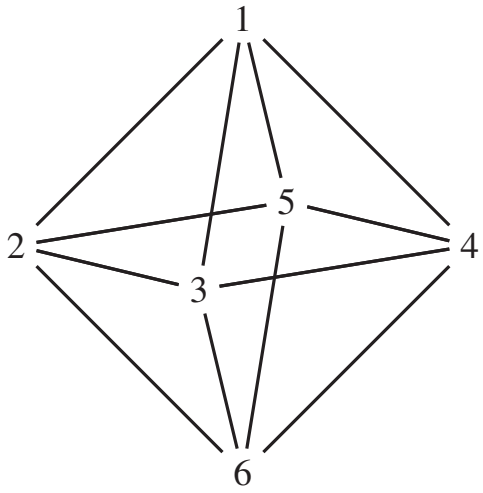
where we can assume that $r \leqslant s \leqslant t$. This time the corresponding equations to those in (1) are

$$\operatorname{lcm}(n_1, n_2) = p^s \times H_1 \qquad \operatorname{lcm}(n_1, n_3) = p^t \times H_2 \qquad \operatorname{lcm}(n_2, n_3) = p^t \times H_3, \tag{2}$$

where $s < t$. Now remember that the same triple w_1, w_2 and w_3 is produced both by the gcd's in (1) and the lcm's in (2). As far as the powers of p are concerned, (1) produces the triple p^a, p^a, p^b (where $a < b$) and (2) produces the triple p^s, p^t, p^t (where $s < t$). But these two triples cannot possibly be the same and so we have the desired contradiction.

b) The answer is No and a possible example follows. Isabelle initially chooses 1, 2, 2 and 2 and Annabelle initially chooses 1, 1, 1 and 2. Each set yields six pairs of numbers so Isabelle has six gcd's and Annabelle has six lcm's. Do the calculations to verify that each writes down the same set of six numbers but that the numbers in this set are not all equal.

4. In a graph a matching is a subset of edges where no two of them are incident to the same vertex. A perfect matching is one where each vertex of the graph is incident to an edge in the set. Let us consider a octahedron (illustrated below). How many perfect matchings are there? How many Hamiltonian cycles are in an octahedron?
*Reminder: A **Hamiltonian cycle** in a graph is a route visiting every vertex exactly once before returning to the starting vertex.*



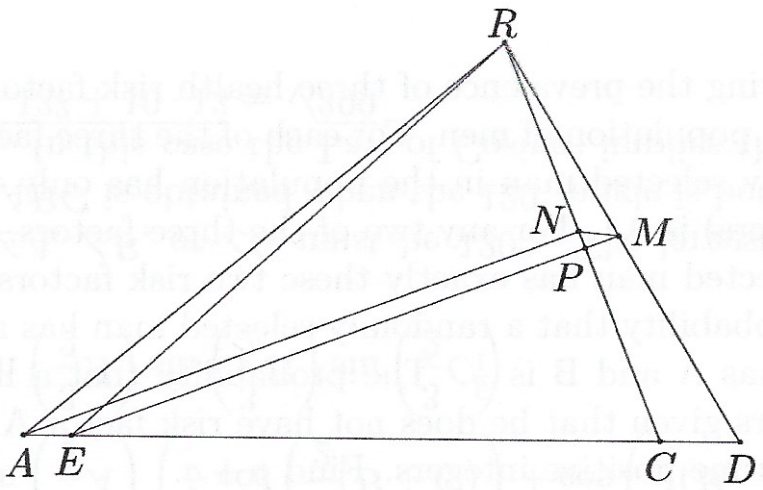
SOLUTION

Given there are 6 vertices in this graph, we expect that a perfect matching will have 3 edges. For each matching only one edge incident to vertex 1 can be used. The same applies to vertex 6. The edges chosen must be incident to two vertices which are adjacent on the equator. Otherwise there will not be an edge to connect the two unconnected vertices. Therefore we have the choice of using one out of the four edges on the equator at a time to connect two vertices on the equator. Then the other two vertices on the equator can each connect to either 1 or 6 at a time. This produces a total of 8 different sets each of which gives us a perfect matching.
As for the Hamiltonian cycles there are two different configurations. We can start from vertex 1 and descend to the equator. Then the route can either continue to another vertex on the equator or directly descend to vertex 6. In the first case there are 2 opposite pairs of edges on the equator to be used, 2 choices to connect to vertex 1 and 2 choices to leave vertex 6. So there are 8 different Hamiltonian cycles in the first case. In the second case there are 4 choices to descend from vertex 1 to the equator and 2 ways to return from vertex 6 to the equator. So the second case also yields 8 different possibilities.
Going from vertex 6 to the equator and onto vertex 1 would only reverse the routes that we have already counted.
So the grand total for the Hamiltonian cycles is 16.

5. In triangle RED, RD has length 1 unit, $\angle DRE = 75°$ and $\angle RED = 45°$. Let M be the midpoint of segment RD. Point C lies on side ED such that RC is perpendicular to EM. Extend segment DE through E to point A such that $CA = AR$. Find AE in the form $\dfrac{a-\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is a positive integer.

SOLUTION

The configuration is illustrated in the diagram below. Since there is a numerical element in the problem, there is no hope of a purely geometrical solution. We can expect to make use of trigonometrical results, such as the sine rule.



The segment EM is shown and recollect that this entails right angles at P. The same is true of the angles at N, where AN is an altitude of the isosceles triangle RAC. Hence AN and EM are parallel. We now have the inviting prospect that the quadrilateral $NAEM$ be a parallelogram. To this end, consider the triangle RCD. The point M is the midpoint of RD (given) and N is the midpoint of RC because AN is an altitude of triangle RAC. Hence by the midpoint theorem, MN is parallel to CD. But by the conditions of the problem, D, C, E and A are collinear and so MN is parallel to EA and $NAEM$, having two pairs of opposite sides parallel, is indeed a parallelogram.

We have to find AE: before examining any angles, note that the midpoint theorem also gives information about lengths. It implies that $2MN = CD$. It follows immediately that

$$2AE = CD \quad \text{or} \quad AE = \frac{1}{2}CD. \tag{1}$$

Now we must consider the angles that we are given. Since there are a host of them, it is strongly recommended that you make a *large* copy of the diagram on which to enter their values. Let us focus on triangle RCD because finding CD will give us AE. In the adjacent triangle PEC, denote $\angle PEC$ by x. This immediately implies that $\angle REM = 45° - x$. Also since triangle PEC is right-angled at P, $\angle ECP = 90° - x$. In triangle RCD, $\angle CDR = 60°$. (This follows from the other two angles of triangle RED.) Since $\angle ECP = \angle ECR$ is an exterior angle of triangle RCD, $\angle DRC = 90° - x - 60° = 30° - x$.

Now it is time to do some trigonometry: first recollect that $RM = MD$, so involving these segments could be productive. Applying the sine rule to triangles REM and EMD yields

$$\frac{RM}{EM} = \frac{\sin(45° - x)}{\sin 75°} \qquad \text{and} \qquad \frac{EM}{MD} = \frac{\sin 60°}{\sin x}. \tag{2}$$

Multiplying these two equations together gives

$$\frac{RM}{EM}\times\frac{EM}{MD}=1=\frac{\sin(45^\circ-x)}{\sin 75^\circ}\times\frac{\sin 60^\circ}{\sin x}=\frac{\sin(45^\circ-x)}{\sin(45^\circ+30^\circ)}\times\frac{\sin 60^\circ}{\sin x}. \tag{3}$$

(Note that here the first equality follows because $RM = MD$.) Equation (3) can be simplified as

$$1=\frac{\frac{1}{\sqrt{2}}(\cos x-\sin x)\sin 60^\circ}{\frac{1}{\sqrt{2}}(\cos 30^\circ+\sin 30^\circ)\sin x}=\frac{\sqrt{3}(\cos x-\sin x)}{(\sqrt{3}+1)\sin x}. \tag{4}$$

Now dividing top and bottom of (4) by $\sin x$ and rearranging gives

$$\cot x=\frac{2\sqrt{3}+1}{\sqrt{3}}. \tag{5}$$

(It is a good idea to check the details here.)

Now return to triangle RCD and apply the sine rule. This gives

$$CD=\frac{CD}{RD}=\frac{\sin(30^\circ-x)}{\sin(90^\circ+x)}=\frac{\cos x-\sqrt{3}\sin x}{2\cos x}=\frac{1-\sqrt{3}\tan x}{2}. \tag{6}$$

We can now use (5) to calculate $\tan x$ as

$$\frac{\sqrt{3}}{2\sqrt{3}+1}=\frac{\sqrt{3}(2\sqrt{3}-1)}{11}=\frac{6-\sqrt{3}}{11}.$$

Substituting in (6) gives

$$CD=\frac{1-\frac{\sqrt{3}(6-\sqrt{3})}{11}}{2}=\frac{14-6\sqrt{3}}{22}.$$

Finally, use (1) to obtain

$$AE=\frac{1}{2}CD=\frac{14-6\sqrt{3}}{44}=\frac{7-3\sqrt{3}}{22}=\frac{7-\sqrt{27}}{22},$$

in the required form.

6. Consider a sequence of four non-negative integers. A new sequence of non-negative integers is formed from it as follows. Replace each integer by the absolute value of the difference between it and its successor, treating the first integer as the successor of the fourth. For example, 1, 0, 1, 0 produces 1, 1, 1, 1.

a) In this first part all the integers are either 0 or 1. For any such sequence can you, by repeating the process described above, arrive at the sequence 0, 0, 0, 0?

b) Now suppose you have a sequence of any four non-negative integers to which you apply the same process. Can you always arrive at the sequence 0, 0, 0, 0?

c) Can you always arrive at the sequence 0, 0, 0, 0, 0 if you start with a sequence of five non-negative integers?

In each part you will need to justify your answer fully.

Solution

a) You might want to start by exploring some particular cases in order to get a feel for the situation. Using 0 and 1 produces only sixteen distinct sequences of four non-negative integers.

First, 0, 0, 0, 0 is in the required form and 1, 1, 1, 1 becomes 0, 0, 0, 0 after one step.

Now consider the sequences with three 0's and one 1. For example,

$$1, 0, 0, 0 \to 1, 0, 0, 1 \to 1, 0, 1, 0 \to 1, 1, 1, 1 \to 0, 0, 0, 0.$$

You may begin to suspect that the process arrives at 0, 0, 0, 0 in every case but, of course a general proof is required. Shifting our viewpoint, let us analyse the behaviour of a particular term, a_n say, in the sequence. Since we are working with binary numbers adding and subtracting are the same. (Ensure that you understand this statement, recollecting that both 0 and 1 are self-inverse under addition.) For simplicity, we will use addition. Consider what happens to the term a_n.

a_n
$\to a_n + a_{n+1}$
$\to a_n + a_{n+1} + a_{n+1} + a_{n+2}$
$\to a_n + a_{n+1} + a_{n+1} + a_{n+2} + a_{n+1} + a_{n+2} + a_{n+2} + a_{n+3} = a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3}$
$\to a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3} + a_{n+1} + 3a_{n+2} + 3a_{n+3} + a_{n+4}$ (where $a_n = a_{n+4}$)
$= 2a_n + 4a_{n+1} + 6a_{n+2} + 4a_{n+3}$.

Repeating this for all terms, we see that, after four iterations, all numbers in the sequence will be even. Because we are using binary arithmetic, we have arrived at 0, 0, 0, 0.

Note that we may arrive at 0, 0, 0, 0 in fewer than four iterations but, since it will stay that way, we can be confident that we have 0, 0, 0, 0 when there have been four iterations. Also, with part b) in mind, note that 0 is the even binary number.

b) Now we consider a sequence of any four non-negative integers. Again, an initial example will make the general argument clearer. Apply the process to 31, 20, 45, 62.

31	20	45	62
11	25	17	31
14	8	14	20
2×7	2×4	2×7	2×10
2×3	2×3	2×3	2×3
2×0	2×0	2×0	2×0
0	0	0	0

Two things are clear about the table. First, the terms become all even in row 3. Second, from line 4 we can effectively start again with smaller numbers, So what happens in general?
We do not, as in part *a)*, have the convenient feature that addition and subtraction are the same. However, the analysis analogous to that of part *a)* starts as follows.

$$a_n \to |a_n - a_{n+1}| \to ||a_n - a_{n+1}| - |a_{n+1} - a_{n+2}||. \tag{1}$$

This is a lot messier than before but the message of part *a)* is that we are essentially interested in the ***parity*** of any terms we encounter. (Recollect that the ***parity*** of any integer refers to whether it is odd or even.)

Let us use the notation $P_y(x)$ for the parity of the integer x. We can then write down the following useful identities for any pair of integers a and b,

$$P_y(a + b) = P_y(a - b) = P_y(|a - b|). \tag{2}$$

The various cases that arise can easily be checked. You can probably do this mentally but an example follows. Suppose a is odd and b is even so that they can be expressed as $a = 2\lambda + 1$ and $b = 2\mu$, where λ and μ are integers. We now have:
$P_y(a + b) = P_y(2\lambda + 2\mu + 1) = P_y(2(\lambda + \mu) + 1)$ (odd);
$P_y(a - b) = P_y(2\lambda - 2\mu + 1) = P_y(2(\lambda - \mu) + 1)$ (odd);
$P_y(|a - b|) = P_y(a - b)$ or $P_y(b - a)$ (odd).

We can further remark that the way parities combine under addition follows exactly the same pattern as binary addition, with odd numbers corresponding to 1 and even numbers to 0. If we label parities in this way, then (2) can be expanded to

$$P_y(a + b) = P_y(a - b) = P_y(|a - b|) = P_y(a) + P_y(b). \tag{3}$$

Following the previous analysis, consider the parities in the second iteration (1). Using (3), we can write this as follows:

$$\begin{aligned} & P_y(||a_n - a_{n+1}| - |a_{n+1} - a_{n+2}||) \\ = \; & P_y(|a_n - a_{n+1}| + |a_{n+1} - a_{n+2}|) \\ = \; & P_y(|a_n - a_{n+1}|) + P_y(|a_{n+1} - a_{n+2}|) \\ = \; & P_y(a_n + a_{n+1}) + P_y(a_{n+1} + a_{n+2}) \\ = \; & P_y(a_n) + P_y(a_{n+1}) + P_y(a_{n+1}) + P_y(a_{n+2}). \end{aligned}$$

If the foregoing looks a bit strange, remember that the parities are now all written in terms of binary digits. However, it shows that the whole procedure will follow the same pattern as in part *a)* and so we know that at the fourth iteration all numbers will be even. We can then take out a factor of 2 and work with a set of numbers which are smaller than before. The process can then be repeated as many times as necessary. At each stage we have smaller numbers than before and they are all even. We conclude that 0,0,0,0 is always possible.

c) In this case some sequences terminate in 0, 0, 0, 0, 0 and some do not. Obviously the sequence 1, 1, 1, 1, 1 does but there are some sequences that repeat after the process has been applied a finite number of times. One such is 1, 1, 1, 1, 0.

7. Alia, Bella and Catherine are multiplying fractions, aiming to obtain integers. Each of them can multiply as many fractions as she likes (including just one) and can use the same fraction more than once.
Alia's fractions are of the form $\dfrac{n+1}{n}$, where n is a positive integer.
Bella's fractions are of the form $\dfrac{6p-5}{3p+6}$, where p is a positive integer.
Catherine's fractions are of the form $\dfrac{4q-1}{2q+1}$, where q is a positive integer.
Which integers can each of them obtain?

[Mathematical Olympiad for Girls 2016 Question 5]

SOLUTION

Taking $n = 1$, Alia gets the integer 2. If $N > 2$, then Alia can obtain N as $\dfrac{2}{1} \times \dfrac{3}{2} \times \cdots \times \dfrac{N}{N+1}$. Thus Alia can obtain any integer except 1. (She cannot obtain 1 because all her fractions are greater than 1.)

Bella, however, cannot obtain any positive integers. Her store consists of fractions of the form $\frac{N}{3D}$, where the positive integer N is not a multiple of 3. If she multiplies m of these fractions together, she will obtain a fraction of the form $\frac{X}{3^m Y}$, where the positive integer X is not a multiple of 3. Since the term 3^m cannot cancel, this will never be a positive integer.

Since the fractions available to Catherine have odd numerators and odd denominators, the same will be true of any fraction she can create by multiplying them. Therefore she certainly cannot reach any even positive integer. She can, however, reach any odd positive integer.

This may be seen as follows. let $f(q)$ be the fraction $\frac{4q-1}{2q+1}$.

Then $f(1) = 1$, $f(4) \times f(7) = \frac{15}{9} \times \frac{27}{15} = 3$ and $3 \times f(4) = 3 \times \frac{15}{9} = 5$. Thus Catherine can obtain 1, 3 and 5.

Suppose now that $m \geqslant 2$ and that Catherine has managed to obtain all odd numbers up to and including $4m - 3$. We know that she can do this for $m = 2$. Since $2m + 1 < 4m - 3$, she can obtain $2m + 1$ and so she can do the following calculations:

$$4m - 1 = (2m + 1) \times \frac{4m - 1}{2m + 1} = (2m + 1) \times f(m),$$

$$4m + 1 = (2m + 1) \times \frac{12m + 3}{6m + 3} = (2m + 1) \times f(3m + 1).$$

She has now added $4m - 1$ and $4m + 1$ to her list and has extended it to reach all the odd numbers up to and including $4m + 1 = 4(m + 1) - 3$.

This means that Catherine, having obtained 5, can obtain 7 and 9, then 11 and 13 and so on. Hence Catherine can obtain any odd positive integer.

8. In Hanna Neumann, Sheet 2, Question 8, you may have used a Renaissance method to solve a particular type of cubic equation. At that time such equations were sorted into different types because mathematicians were still not entirely at ease with the notion of negative numbers. This question explores a general method for solving any cubic equation. We start from the equation

$$ax^3 + bx^2 + cx + d = 0, \qquad (*)$$

where $a \neq 0$, b, c and d are real constants and x is real.

a) Express x in terms of another real variable y so that equation (*) has no term in x^2. Then further reduce the equation so that the leading coefficient is 1.

b) Then equation now has the form

$$y^3 + ey + f = 0, \qquad (1)$$

where e and f are real constants. To (1), apply the substitution $y = z + \dfrac{s}{z}$, where s is a real constant, at present undefined.

c) Find an expression for s in terms of e that converts (1) into a quadratic equation in z^3.

d) Explain how the roots of the equation found in part *c)* can be used to find the solutions of the original cubic (*).

e) Use the method explored here to solve the equation

$$x^3 - 9x^2 + 36x - 80 = 0. \qquad (2)$$

SOLUTION

a) The simplest type of substitution would be a linear one; to keep things as simple as possible let us see if we can get away with a substitution of the form $x = y - k$. Equation (*) becomes

$$a(y-k)^3 + b(y-k)^2 + c(y-k) + d = 0. \qquad (3)$$

The term in y^2 has coefficient $(-3ak + b)$, so we need to take $k = \dfrac{b}{3a}$. Then, on making the substitution and collecting like terms, (3) becomes

$$ay^3 + b\left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0.$$

To convert this to the desired form in (1), divide by $a \neq 0$ to obtain

$$y^3 + \frac{1}{a}\left(c - \frac{b^2}{3a}\right)y + \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0. \qquad (4)$$

We now have an equation of the desired form where

$$e = \frac{1}{a}\left(c - \frac{b^2}{3a}\right) \qquad \text{and} \qquad f = \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right). \qquad (5)$$

b) Making the substitution $y = z + \dfrac{s}{z}$ in (1) yields

$$\left(z + \frac{s}{z}\right)^3 + e\left(z + \frac{s}{z}\right) + f = 0.$$

Now multiply out and multiply up by z^3 to obtain

$$z^6 + (3s + e)z^4 + fz^3 + s(3s + e)z^2 + s^3 = 0. \tag{6}$$

c) In (6) note that the coefficients of both z^4 and z^2 are divisible by $(3s + e)$. Further, to obtain a quadratic in z^3, this expression must be 0. Consequently we must take $s = -\frac{e}{3}$. Then (6) becomes

$$z^6 + fz^3 + s^3 = 0. \tag{7}$$

d) Once (7) is solved, it is possible to track back through the substitutions and eventually obtain a solution of the original cubic in x. This process will be illustrated in part *e)* by the solution of (2).

e) Substituting $x = y - k$ in equation (2), we have

$$(y - k)^3 - 9(y - k)^2 + 36(y - k) - 80 = 0 \tag{8}$$

Inspection shows that the coefficient of y^2 is 0 if $k = -3$. Substituting this value in (8) yields

$$y^3 + (27 - 54 + 36)y + (27 - 81 + 108 - 80) = 0$$

or

$$y^3 + 9y - 26 = 0. \tag{9}$$

Thus, in the notation of (1), $e = 9$ and $f = -26$. Now it is time to make the substitution $y = z + \frac{s}{z}$ and the discussion in part *c)* indicates that $s = \frac{-e}{3} = -3$. Then by (7), the required quadratic in z^3 is $z^6 - 26z^3 - 27 = 0$. This factorises as $(z^3 - 27)(z^3 + 1) = 0$ and so $z^3 = 27$ or $z^3 = -1$. It follows immediately that $z = 3$ or $z = -1$.

Recalling that $y = z + \frac{s}{z} = z + \frac{(-3)}{z}$, we see that $z = 3$ implies $y = 2$ and $x = y - k = y + 3 = 5$. For the other value, $z = -1$ again implies $y = 2$ and $x = 5$. We conclude that the cubic equation (2) has a single real root and that it is $x = 5$.

A final comment: The solution above shows that $x^3 - 9x^2 + 36x - 80$ has a factor $x - 5$. Carrying out the factorisation, we have

$$x^3 - 9x^2 + 36x - 80 = (x - 5)(x^2 - 4x + 16).$$

Is there another value of x that makes $x^2 - 4x + 16$ equal to zero? No, because the discriminant of this quadratic is $16 - 4 \times 16$, which is negative. This confirms that $x = 5$ is the only real root of the original cubic equation.

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Srinivasa Ramanujan

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more information.

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Version 2.0, Mar 2021

1. Sir Cumference, a portly knight, is on a quest to slay a most aggravating hundred-headed dragon. Unfortunately, he is having trouble with his magic sword and all of the local sword-fixing wizards are either on holiday or working for more influential knights. Due to the malfunction, Sir Cumference is limited to cutting off 15, 17, 20 or 5 heads with a single blow of his sword. If he cuts 15 heads from the dragon, then 24 heads immediately grow back. Similarly, if he cuts off 17, 20 or 5 heads, then 2, 14 or 17 heads will grow back, respectively. The dragon dies if it has no heads left. Can Sir Cumference kill the dragon?

SOLUTION

However many heads Sir Cumference manages to remove, the difference between his score and the number of heads that grow back is always a multiple of 3. This suggests that it might be useful to work modulo 3. Initially the dragon has 100 heads and $100 \equiv 1 \pmod 3$. Each blow of the sword changes the number of heads by a multiple of 3 and so the number of heads remains congruent to 1 modulo 3. Thus the dragon never loses all its heads and so Sir Cumference never manages to kill it.

2. In Autumn 2012, the school where Ms Z teaches mathematics launched a fundraising campaign to celebrate the 600^{th} anniversary of its foundation. Find the date of the school's foundation from the following information. The actual date of this anniversary lies in the period from 2013 to 2099 inclusive. The anniversary year is the product of a prime and the square of a different prime. The two primes involved have the same units digit, which also happens to be the digit sum of the anniversary year.

SOLUTION

There are a great many possible years and so there is a role for algebra in reducing the number of cases involved. Suppose that the integer corresponding to the anniversary year is equal to pq^2, for p and q distinct primes. Now p and q have the same final digit, which is also equal to the digit sum of the anniversary year. We will initially focus on finding what this digit might be. Since $2013 \leqslant pq^2 \leqslant 2099$, pq^2 has thousands digit equal to 2. Furthermore, either the tens digit is 1 and the units digit is at least 3 (for 2013 to 2019) or the tens digit is at least 2 (2020 onwards). Thus pq^2 has a digit sum that is at least 4. This digit sum is the last digit of both p and q, so the possibilities are 5, 7 and 9. (Check that you understand why even digits are excluded.)

If the digit sum is 5, then p and q are distinct primes ending in 5. However 5 itself is the only prime ending in 5 and so this case is impossible.

If the digit sum is 9, then then p and q are distinct primes ending in 9. The smallest such prime is 19 and so $pq^2 > 19^3 = 6859 > 2099$. Hence the digit sum cannot be 9.

It follows that the digit sum is 7 and so p and q are distinct primes ending in 7. If $q > 17$, then $q \geqslant 37$ and $pq^2 \geqslant 7 \times 37^2 = 9583 > 2099$. It follows that q must be 7 or 17. If $p > 37$, then $p \geqslant 47$ and $pq^2 \geqslant 47 \times 7^2 = 2303 > 2099$. This also is impossible so p is 7, 17 or 37. Since $p \neq q$, there are just four possibilities to try, which may be conveniently displayed in a table.

q	p	pq^2	Comment
7	17	833	Too small
7	37	1813	Too small
17	7	2023	In correct range
17	37	10693	Too large

Thus just one pair gives a value of pq^2 in the correct range and we observe that the digit sum of 2023 is $2 + 0 + 2 + 3 = 7$, as required. Thus the anniversary year is 2023 and the school was founded in 1423.

3. This question concerns a triangle ABC.

a) Prove that the internal angle bisector of angle A and the perpendicular bisector of BC meet on the circumcircle of the triangle.

b) Prove that the the internal and external angle bisectors of A pass through the ends of the diameter of the circumcircle that is perpendicular to BC.

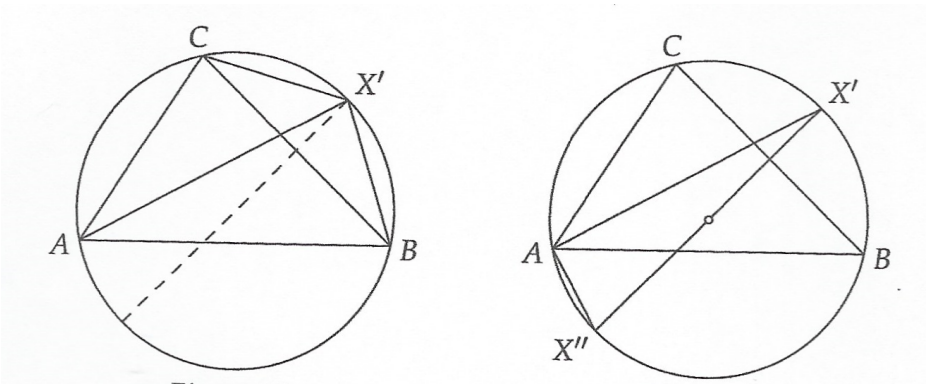
c) Let X' be the point where the the internal angle bisector of angle A meets the circumcircle of the triangle. Further, let Y' and Z' be the points where the the internal angle bisectors of angles B and C meet the circumcircle. Prove that the incentre I of triangle ABC is the orthocentre of triangle $X'Y'Z'$.

Note 1: Altitudes of a triangle appear in Srinivasa Ramanujan Sheet 1, Question 3 and Sheet 4, Question 7. The three altitudes meet in a point, which is called the ***orthocentre*** *of the triangle.*

Note 2: The internal angle bisectors of the angles of a triangle meet in a point, which is called the ***incentre*** *of the triangle.*

SOLUTION

The two diagrams below refer to the first two parts of the question

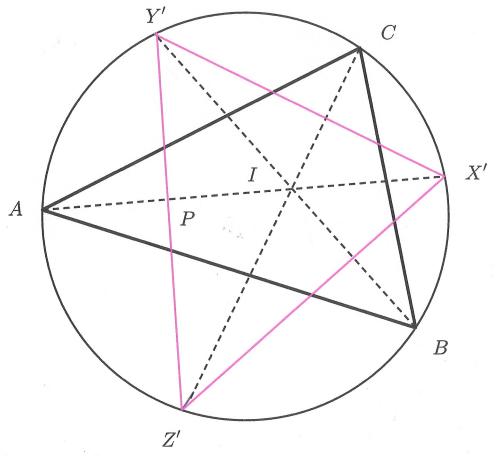


a) The left-hand diagram shows the triangle ABC and its circumcircle. The internal angle bisector of angle A is shown meeting the circumcircle at X'. Because $\angle BAX' = \angle X'AC$, the chords BX' and $X'C$ are equal. Therefore triangle $BX'C$ is isosceles and the perpendicular bisector of BC (shown as a dotted line) passes through X'. This proves the first result.

b) The circumcentre of triangle ABC is the point where the perpendicular bisectors of the sides of the triangle meet. It follows that the dotted line in the left-hand diagram passes through the circumcentre of triangle ABC; suppose that this perpendicular bisector meets the circumcircle again at X''. This line segment is shown again in the right-hand diagram and the circumcentre is marked with a small circle. We see immediately that $X'X''$ is a diameter of the circumcircle

and so $X'AX'' = 90°$. Since AX' is the internal bisector of the angle at A, AX'' must be the external bisector of this angle. This proves the required result.

c) The diagram below shows the triangle $X'Y'Z'$.



By angles in the same segment, we have

$$\angle AX'Y' = \angle ABY' = \frac{1}{2}\angle B \text{ and } \angle X'Y'B = \angle X'AB = \frac{1}{2}\angle A \text{ and } \angle BY'Z' = \angle BCZ' = \frac{1}{2}\angle C.$$

Let P be the intersection of AX' and $Y'Z'$. From the equations above it follows that

$$\angle Y'PX' = 180° - \frac{1}{2}(\angle A + \angle B + \angle C) = 90°.$$

Hence $X'A$ is an altitude of triangle $X'Y'Z'$ and the same is true of $Y'B$ and $Z'C$. But we already know that these line segments meet in the incentre, so this point is the orthocentre of $X'Y'Z'$, as required.

4. Solve the following system of equations, where $n \geqslant 2$ and all the x_i are real.

$$\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1} = \cdots = \sqrt{x_n} + \sqrt{x_1 + \cdots + x_{n-1}};$$

$$x_1 - x_2 = 1.$$

SOLUTION

First consider the equation

$$\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1},$$

where $n \geqslant 3$. On squaring both sides, we obtain

$$x_1 + 2\sqrt{x_1}\sqrt{x_2 + \cdots + x_n} + x_2 + \cdots + x_n = x_2 + 2\sqrt{x_2}\sqrt{x_3 + \cdots + x_n + x_1} + x_3 + \cdots + x_n + x_1.$$

Now subtract $x_1 + \cdots + x_n$ from both sides and cancel a factor of 2 to obtain

$$\sqrt{x_1}\sqrt{x_2 + \cdots + x_n} = \sqrt{x_2}\sqrt{x_3 + \cdots + x_n + x_1}.$$

On squaring both sides again, we have

$$x_1(x_2 + \cdots + x_n) = x_2(x_3 + \cdots + x_n + x_1).$$

A term x_1x_2 may now be subtracted from both sides to leave

$$x_1(x_3 + \cdots + x_n) = x_2(x_3 + \cdots + x_n).$$

This in turn may be rewritten as $(x_1 - x_2)(x_3 + \cdots + x_n) = 0$. But the last of the original equations states that $(x_1 - x_2) = 1$, so we have $x_3 + \cdots + x_n = 0$.

Now note that our equations contain the square roots of x_3, $\cdots$, x_n, so all these terms are non-negative. However, we have just seen that that their sum is 0 and conclude that they are individually equal to 0.

Moving on to find the value of x_2, consider

$$\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} = \sqrt{x_3} + \sqrt{x_4 + \cdots + x_n + x_1 + x_2}.$$

If we square once more and use $x_3 = x_4 = \cdots = x_n = 0$, then we get

$$x_1 + 2\sqrt{x_1}\sqrt{x_2} + x_2 = x_1 + x_2.$$

This in turn implies $x_1x_2 = 0$. Hence either $x_1 = 0$ or $x_2 = 0$. But now recall that $x_1 - x_2 = 1$, so we are forced to choose $x_1 = 1$ and $x_2 = 0$. (Why are the values this way round?) Therefore the full solution for $n \geqslant 3$ is

$$x_1 = 1, \quad x_2 = x_3 = \cdots = x_n = 0.$$

It remains to discover what happens when $n = 2$. In this case the equations reduce to

$$\sqrt{x_1} + \sqrt{x_2} = \sqrt{x_2} + \sqrt{x_1} \qquad \text{and} \qquad x_1 - x_2 = 1.$$

The first equation is satisfied for all non-negative values of x_1 and x_2. Then, with the condition imposed by the second equation, we see that the full solution is

$$x_2 = k \ (k \text{ is any non-negative real}) \qquad \text{and} \qquad x_1 = 1 + k.$$

Finally observe that a lot of squaring has happened. It would therefore be advisable to check that all the solutions are valid and this can easily be done.

5. The school where Mr Z (Ms Z's brother) teaches philosophy celebrates an anniversary in 2019. Mr Z plans to invite 2019 philosophers and mathematicians to a conference. Each conference member has an assigned seat in the new lecture theatre. The first member to enter is deep in thought and sits in a seat uniformly at random. After this, the members enter one at a time and in turn sit in their assigned seat if it is free and in a uniformly chosen unoccupied seat if the assigned seat is already taken. Find the probability that the final member gets their assigned seat.

Solution

Assume that the conference members are numbered 1, 2, ..., 2019 and that they enter the lecture theatre in ascending order. Note that, after k members have entered, then either one or none of the seats numbered $k + 1$, $k + 2$, ..., 2019 is already occupied. If none of these seats is occupied, then everyone from person $k + 1$ onwards gets their assigned seat.

The probability that the absent-minded first arrival sits in the wrong seat is 2018/2019. Then for $k \geqslant 1$, suppose that one of the remaining seats $k + 1$, $k + 2$, ..., 2019 is wrongly occupied

after the k^{th} arrival. Then after the $(k+1)^{\text{st}}$, one of the seats $k+2$, , 2019 must also be wrongly occupied ***unless*** the $(k+1)^{\text{st}}$ member arrived, found their seat occupied and sat in the ***unique*** seat numbered 1, 2, ..., k not currently occupied. This event has probability

$$\frac{1}{2019-k} \times \frac{1}{2019-k}.$$

Thus the probability that it ***did not happen*** (that is, there continued to be one wrongly-occupied seat) is

$$1 - \frac{1}{(2019-k)^2} = \frac{(2019-k)^2-1}{(2019-k)^2} = \frac{(2019-k-1)(2019-k+1)}{(2019-k)^2}.$$

So the probability that there is always a seat wrongly occupied (including seat 2019 when it is time for the last arrival to enter) is

$$\frac{2018}{2019} \times \frac{2019 \times 2017}{2018^2} \times \frac{2018 \times 2016}{2017^2} \times \cdots \times \frac{3 \times 1}{2^2}.$$

But we can cancel all the terms in this giant fraction apart from one of the 2's in the last denominator. Hence the required probability is 1/2.

Alternative method: You may well have calculated the probabilities step by step, as above. However, there is a subtler and shorter approach to the problem. Observe that the fate of the last person (number 2019) is decided as soon as either the first or the last seat is selected. This is because the last person gets either the first seat or the last. Any other seat will necessarily be selected by the time the last conference member arrives. At each step the first seat or the last is equally likely to be taken. Hence the last person, number 2019, gets seat 2019 with probability 1/2. If you find this solution hard to follow, try checking the possible steps with a very much smaller "conference", for example one with four members.

6. *a)* Mrs M is a maths teacher who is taking a party of pupils on an end-of-term trip to the local zoo. Her school has a healthy eating policy, so Mrs M, despite her passion for saving money, is obliged to patronise the rather superior chicken nugget outlet at the zoo. The nuggets are sold in small packs of 5 and large packs of 11. Mrs M wishes to make her purchase as economical as possible, so she is interested in the largest positive integer which cannot be expressed as the sum of a multiple of 5 and a multiple of 11. Help Mrs M by calculating this number for her.
Note 1 on vocabulary: Mrs M wants to know the largest positive integer that cannot be expressed as a ***linear combination*** *of 5 and 11.*
Note 2 (more frivolous): M does not stand for Maths; what might it stand for?

b) If a and b are coprime positive integers, then prove that there is a positive integer K such that every N greater than K can be written in the form $N = ja + kb$, where the coefficients j and k are non-negative integers.

SOLUTION

a) One way to help Mrs M is to construct a five-column table, as shown below.

1	2	3	4	5
6	7	8	9	2×5
11	12	13	14	3×5
$11+5$	17	18	19	4×5
$11+2\times5$	2×11	23	24	5×5
$11+3\times5$	$2\times11+5$	28	29	6×5
$11+4\times5$	$2\times11+2\times5$	3×11	34	7×5
$11+5\times5$	$2\times11+3\times5$	$3\times11+5$	39	8×5
$11+6\times5$	$2\times11+4\times5$	$3\times11+2\times5$	4×11	9×5

As we can see, some of the integers involved are written as linear combinations of 5 and 11. The largest number that is not in this form is 39. We can see that 39 is the largest number because subsequent rows of the table can be obtained by adding one more multiple of 5 to each of the entries in the row above.
Remark: Important features of the table are the positions of the multiples of 11. This particular example may seem suspiciously tidy, with these multiples marching across the table from left to right. Does this always happen? No: for example, you could check this by constructing the three-column table produced by replacing 5 by 3 and 11 by 8.

b) The method of *a)* suggests a method of proof in this general case. We shall assume that $a < b$ and we shall consider a table with a columns. If you considered a second example in *a)*, then you may have noticed that, though the multiples of the larger integer appeared in a somewhat haphazard order, they are in different columns of the table. If this is true in general, then we have a solution when the first $a-1$ multiples of b have appeared in the first $a-1$ columns of the table. We do not have to consider the ath column because this contains multiples of a only.

We argue by contradiction. Suppose there are distinct integers m and n, both less than a, such that mb and nb both appear in the same column of the table. (For convenience assume that $m < n$.) The way the table is constructed means that these two integers share the same remainder r, say, on division by a. We can write $mb = ak_m + r$ and $nb = ak_n + r$. It follows that $(n-m)b = a(k_n - k_m)$. Therefore $(n-m)b$ is divisible by a and, because a and b are coprime, it must be the case that $n-m$ is divisible by a. But this is a ***contradiction***, since m and n are both less than a. Thus the general result is proved.

The method of proof also shows how K may be expressed in terms of a and b. We note the appearance of $(a-1)b$ in a column of the table (and we may further observe that the column number depends on the remainder of $(a-1)b$ on division by a). All smaller multiples of b will have appeared already, so K will be the entry immediately above $(a-1)b$, namely $(a-1)b - a = ab - a - b$. Checking with Mrs M's table in part *a),* we see that the relevant value of K is $5\times11 - 5 - 11 = 39$.

7. It is well known that, for each positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

and so is a square. Determine whether or not there is a positive integer m such that

$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3$$

is a square.

[British Mathematical Olympiad 2018 Round 2 Question 3]

SOLUTION

First note that the given expression, in factorised form, is

$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3 = \frac{(2m)^2(2m+1)^2}{4} - \frac{m^2(m+1)^2}{4} = \frac{m^2}{4}(5m+3)(3m+1).$$

Now, using the usual abbreviation gcd for greatest common divisor, we have

$$\gcd(5m+3, 3m+1) = \gcd(3m+1, 2m+2) = \gcd(2m+2, m-1) = \gcd(m-1, 4).$$

The only possible values for the last gcd are 1, 2 or 4. Therefore $5m+3$ and $3m+1$ are either both perfect squares or both twice a perfect square (the only possible prime factor they can have in common is 2).

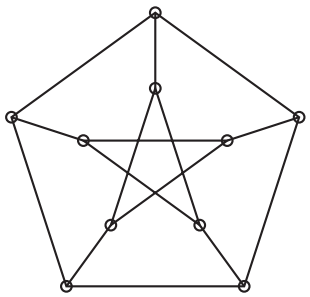
Now $5m+3 \equiv 3 \pmod 5$ and the squares modulo 5 are 0, 1 and 4. Hence $5m+3$ is not a perfect square. Also $3m+1 \equiv 1 \pmod 3$ and the squares modulo 3 are 0 and 1. Thus twice a square modulo 3 is 0 or 2. Hence $3m+1$ is not twice a perfect square. Therefore it is impossible for $(5m+3)(3m+1)$ to be a perfect square.

8. Previous questions have discussed basic graph concepts, such as vertices, edges, faces, adjacency and incidence. Look these up if necessary. This question will explore some more complex situations and will involve the idea of a ***regular graph***. This is is a graph where all the vertices are of the same degree.

a) Let Ω be a be a fixed set with v elements. Define graph $J(v, k, i)$ as follows. The vertices of $J(v, k, i)$ are the subsets of Ω with k elements, where two subsets are adjacent if their intersection contains i elements. Show that $J(v, k, i)$ has $\binom{v}{k}$ many vertices, and it is a regular graph of degree

$$\binom{k}{i}\binom{v-k}{k-i}.$$

b) When $i = 0$, the graphs $J(v, k, 0)$, are known as the Kneser graphs. In particular $J(5, 2, 0)$ is known as the Petersen graph and it is shown in the diagram below.



Explain why, with the given values of $v = 5, k = 2$ and $i = 0$, the graph can be illustrated by this diagram. Then either find a Hamiltonian cycle for this graph or prove that one does not exist.

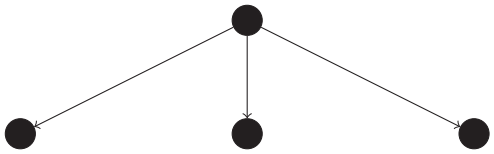
SOLUTION

a) The number of vertices for $J(v, k, i)$ is equal to the number of ways one can choose k elements from a set of v elements. Similarly, each set of k elements can form $\binom{k}{i}$ subsets of i elements. Each set of i elements can be a part of another set of k elements. So now we need to know how many sets of k elements share i elements with the chosen subset of the k-element set we are considering. To do this we have to count the number of ways of choosing $k - i$ (the non-intersecting parts of two sets of k elements) from $v - k$ (the number of elements which have not already been used in the chosen k set) elements. Combining the two binomial coefficients gives the required result.

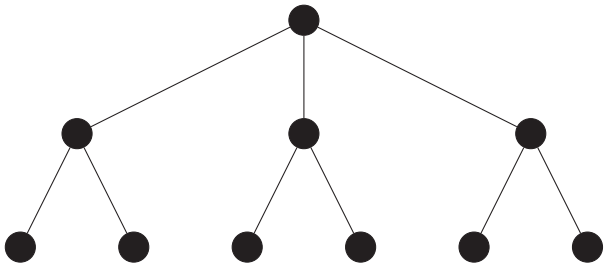
b) The value $v = 5$ tells us there are 5 elements in Ω. The value $k = 2$ suggests that all the vertices in the graph we are interested in correspond to a subset of Ω with 2 elements. From part *a*) we know there are 10 such subsets. Therefore the graph we are constructing will have 10 vertices. The value $i = 0$ implies that in our graph two vertices are adjacent if they correspond to two subsets of Ω with the empty set as their intersection. For a 2-element subset belonging to a set of 5 elements, the subsets with which it does not share an element are provided by the other 3 elements. Now, 3 elements can form 3 distinct subsets of 2 elements. This tells us that the graph we are constructing is regular with vertex degree equal to

3, which is consistent with our find in part *a*). We can use a tree diagram to construct the graph.

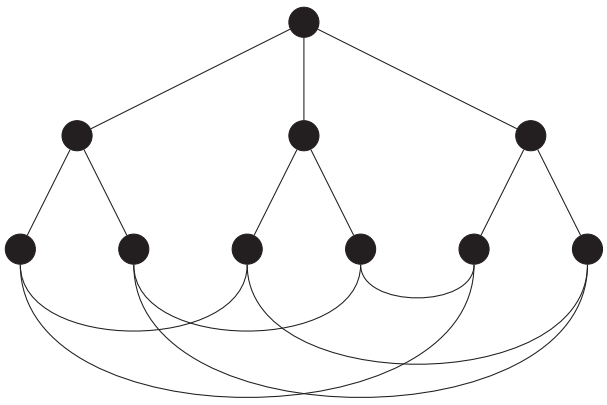
WLOG (a useful abbreviation for the phrase "without loss of generality") we take any vertex as the root and connect it to 3 vertices in the next generation.



The vertices in generation 1 are not adjacent to each other, because we know that they are made of 2 elements out of 3, so they all share one element. In order to retain the degree 3 requirement, we can add another generation.



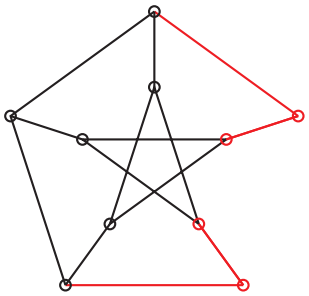
Now we see that all the vertices in the 0^{th} and the 1^{st} generations have degree 3. In the last generation we need to connect each vertex to two others from their own generation. As in the 1^{st} generation, all vertices adjacent to a particular vertex cannot be adjacent to each other. So in order to complete the graph, we have to connect each vertex to descendants of different parents. There is essentially only one way to do this.



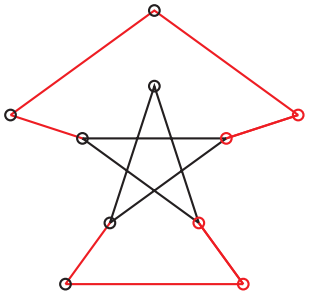
If the previous paragraph seems very mysterious to you, consider the case $\Omega = \{1, 2, 3, 4, 5\}$. Use pairs of these numbers to form the 10 subsets. You can mark 2 numbers in each vertex and connect each vertex to 3 vertices with which they do not share elements. To extend your understanding, draw your completed graph, (perhaps using a drawing package such as Geogebra). See whether you can move the positions of the vertices so that your graph looks

like the one given in the question or the one constructed above.

The Petersen graph has 15 edges and 10 vertices. A Hamiltonian cycle for the Petersen graph would have 10 edges. This means that 5 edges on the Petersen graph are not included in its Hamiltonian cycle. These 5 edges cannot be the 5 edges connecting the outer pentagon and the inner star. Otherwise we have two disjoint subgraphs. This means we can assume that one edge on the outer pentagon is not included in the Hamiltonian cycle. *(See Remark below)* When we remove one edge on the outer pentagon, then there will be two vertices with only two remaining available edges. These edges must be included in the Hamiltonian cycle, so the vertices connected by the removed edge can be included in the Hamiltonian cycle.



As the inner star can have a Hamiltonian path only by the removal of one of its edges, there is no Hamiltonian path for the star with the two vertices reached by red edges as the endpoints. This means the obvious Hamiltonian path visiting all five vertices of the outer pentagon is not entirely in the Hamiltonian cycle we are seeking. This implies that the vertex not yet reached by a red edge must connect to the inner star. By symmetry we can eliminate more edges and choose some necessary edges.

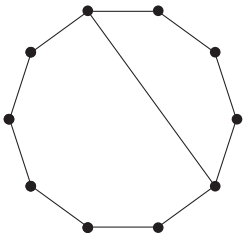


The vertex at the top of the inner star has only two edges remaining at this stage. So they must be included. But this will result in a subcycle within the Hamiltonian cycle we seek. Therefore it is not possible for the Petersen graph to have a Hamiltonian cycle.

Remark 1: The sets of the outer pentagon and the inner star are interchangeable. If you're not sure about this, a drawing package would help you to manipulate your diagram to convince yourself.

Remark 2: You may be interested in this quick method for the last part of the question We can see that every vertex on a Petersen graph is part of 3 cycles of length 5. Each pair of edges it is incident to are in distinct cycles of length 5. Without loss of generality, if a Hamiltonian cycle exists for a Petersen graph, then we can construct a graph in the shape of a decagon which is essentially the same as the Petersen graph, with the perimeter being the Hamiltonian cycle. There are 15 edges on a Petersen graph, so we have 5 edges left to allocate. As the Petersen graph is a regular graph, all its vertices have degree 3. Thus what remains to be done is to assign the 5 edges so that they pair up the 10 vertices.

The decagon graph demonstrates the difficulty of creating 3 cycles of length 5 for each vertex. We are allowed to assign only one edge per vertex. This edge has to be part of 2 cycles of length 5. As we cannot subdivide the 9 other vertices into 2 groups of 4, we cannot create 2 more cycles of length 5. This proves that a Petersen graph cannot be isomorphic to (essentially the same as) a graph with a Hamiltonian cycle of length 10. Therefore there is no Hamiltonian cycle in a Petersen graph.



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Srinivasa Ramanujan

Sheet 8

Solutions and comments

This programme of the Mentoring Scheme is named after Srinivasa Aiyangar Ramanujan (1887–1920)

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Version 1.0, Dec 2019

1. Determine which is larger: $\sqrt[9]{9!}$ or $\sqrt[8]{8!}$

Note: The roots may be clearer as text. They are the ninth root of 9! and the eighth root of 8!

SOLUTION

Denote the ninth root of 9! by u and the eighth root of 8! by v. Then $u^9 = 9!$ and $v^8 = 8!$. Hence $u^{72} = (9!)^8$ and $v^{72} = (8!)^9$. Consequently

$$\left(\frac{u}{v}\right)^{72} = \frac{u^{72}}{v^{72}} = \frac{(9!)^8}{(8!)^9} = \frac{9^8 \times (8!)^8}{(8!)^8 \times 8!} = \frac{9^8}{8!} \geqslant 1.$$

It follows that $\dfrac{u}{v} \geqslant 1$ and so the ninth root of 9! is the larger.

2. *a)* Suppose z_1 and z_2 are complex numbers. Use a geometrical / pictorial argument to show that $|z_1 - z_2| \geqslant |z_1| - |z_2|$. *(Note that reference to a diagram will suffice here.)*

b) Prove that, for any two complex numbers z_1 and z_2, $|z_1 - z_2| \geqslant ||z_1| - |z_2||$.

c) Deduce the triangle inequality for complex numbers, namely $|z_1| + |z_2| \geqslant |z_1 + z_2|$.

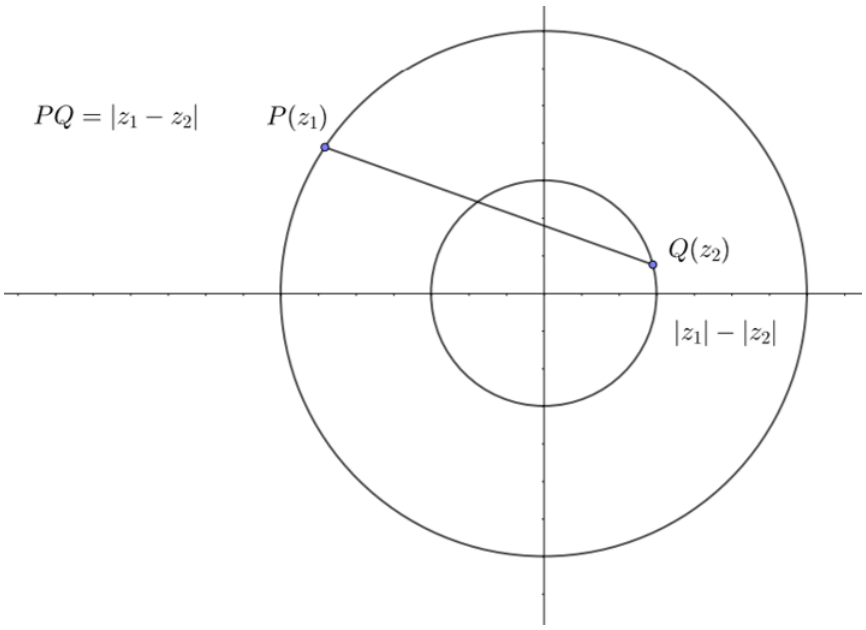
d) Show that

$$\left|\frac{z^2 - 4z + 3}{(z+7)(z-3)}\right| \leqslant \frac{3}{5} \quad \text{for } |z| = 2. \tag{1}$$

e) Find all values of z with $|z| = 2$ such that the inequality in (1) can be replaced by equality.

SOLUTION

a) The diagram below shows part of the of the complex plane, including the points representing the numbers z_1 and z_2. The point P representing z_1 lies on the circle with centre the origin and radius $|z_1|$ and Q representing z_2 lies on the circle with centre the origin and radius $|z_2|$. The diagram illustrates the case $|z_1| > |z_2|$. Since the the distance $PQ = |z_1 - z_2|$, it shows that the smallest value $|z_1 - z_2|$ can take is $|z_1| - |z_2|$.



If $|z_1| = |z_2|$,then $|z_1| - |z_2| = 0$ and the inequality still holds as $|z_1 - z_2| \geqslant 0$. Finally, if $|z_2| > |z_1|$, then we have $|z_1 - z_2| \geqslant 0$ and $|z_1| - |z_2| < 0$. Thus in all cases the inequality holds.

b) By part *a)*, $|z_1 - z_2| \geqslant |z_1| - |z_2|$ and also $|z_2 - z_1| \geqslant |z_2| - |z_1|$. But $|z_2 - z_1| = |z_1 - z_2|$, so we see that $|z_1 - z_2|$ is greater than or equal to both $|z_1| - |z_2|$ and $|z_2| - |z_1|$. However, $||z_1| - |z_2||$ is either equal to $|z_1| - |z_2|$ or is equal to $|z_2| - |z_1|$. Thus we are justified in writing $|z_1 - z_2| \geqslant ||z_1| - |z_2||$.

c) Apply part *a)* to $z_1 + z_2$ and z_2 to obtain $|z_1 + z_2 - z_2| \geqslant |z_1 + z_2| - |z_2|$. This can be rearranged to give $|z_1| + |z_2| \geqslant |z_1 + z_2|$, as required.

d) We have

$$\left|\frac{z^2 - 4z + 3}{(z+7)(z-3)}\right| = \left|\frac{(z-1)(z-3)}{(z+7)(z-3)}\right| = \left|\frac{z-1}{z+7}\right| = \frac{|z-1|}{|z+7|}. \tag{2}$$

By part *c)*, $|z - 1| \leqslant |z| + |-1| = 2 + 1 = 3$.

By part *b)*, $|z + 7| = |z - (-7)| \geqslant ||z| - |-7|| = |2 - 7| = 5$. It follows that $\dfrac{1}{|z+7|} \leqslant \dfrac{1}{5}$.

Substituting these last two results into (2) yields

$$\left|\frac{z^2 - 4z + 3}{(z+7)(z-3)}\right| \leqslant \frac{3}{5} \quad \text{for } |z| = 2.$$

e) It is quite easy to spot that $z = -2$ provides a solution but could there be another one? Suppose that such a solution $z = 2(\cos\theta + i\sin\theta)$ exists.
Then by (2), $5|2(\cos\theta + i\sin\theta) - 1| = 3|2(\cos\theta + i\sin\theta) + 7|$. Squaring both sides yields

$$25|(2\cos\theta - 1) + 2i\sin\theta)|^2 = 9|(2\cos\theta + 7) + 2i\sin\theta)|^2.$$

This is equivalent to

$$25((2\cos\theta - 1)^2 + 4\sin^2\theta) = 9((2\cos\theta + 7)^2 + 4\sin^2\theta).$$

On simplifying this equation becomes

$$25(4 - 4\cos\theta + 1) = 9(4 + 28\cos\theta + 49),$$

which in turn gives

$$125 - 477 = (100 + 252)\cos\theta \quad \text{or} \quad -352 = 352\cos\theta.$$

It follows that $\cos\theta = -1$, so $\theta = \pi$ and $z = -2$. This is the solution noticed before and so we conclude that no other exists.

3. If A, B and C are angles with

$$\sin A + \sin B + \sin C = 0 = \cos A + \cos B + \cos C,$$

prove that

$$\cos 3A + \cos 3B + \cos 3C = 3\cos(A + B + C).$$

SOLUTION

This problem will yield, laboriously, to the application of the appropriate trigonometrical formulae. A more streamlined solution may be obtained by making use of the identity $e^{i\theta} = \cos\theta + i\sin\theta$.

Using this we have $e^{3iA} = \cos 3A + i \sin 3A$, $e^{3iB} = \cos 3B + i \sin 3B$ and $e^{3iC} = \cos 3C + i \sin 3C$. Now terms such as e^{3iA} could come from cubing a sum of exponentials. To simplify the algebra, set $e^{iA} = a$, $e^{iB} = b$ and $e^{iC} = c$. Then the initial conditions imply that

$$a + b + c = \cos A + \cos B + \cos C + i(\sin A + \sin B + \sin C) = 0 + 0i = 0. \qquad (1)$$

Now consider what happens when we cube $a + b + c$. Since (1) shows that both the real and imaginary parts are zero, we are interested in taking out as many factors $(a + b + c)$ as possible. We have

$$\begin{aligned} & (a+b+c)^3 \\ = \; & a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 6abc \\ = \; & a^3 + b^3 + c^3 + 3(a^2 + b^2 + c^2)(a + b + c) - 3a^3 - 3b^3 - 3c^3 + 6abc. \end{aligned}$$

Again using (1), we see that the working above collapses to

$$0 = -2a^3 - 2b^3 - 2c^3 + 6abc, \quad \text{which in turn reduces to } a^3 + b^3 + c^3 = 3abc.$$

It is now time to reinstate the angles and, on doing so, the previous equation becomes

$$e^{3iA} + e^{3iB} + e^{3iC} = 3e^{iA}e^{iB}e^{iC} = 3e^{i(A+B+C)}.$$

Equating the real parts from each side, we have the desired result

$$\cos 3A + \cos 3B + \cos 3C = 3\cos(A + B + C).$$

Remark: You will have noticed that there is a parallel result for sines.

4. The terms of the sequence $\{a_n\}$ defined by $a_{n+2} = \dfrac{a_n + 1887}{1 + a_{n+1}}$ (for $n \geqslant 1$) are positive integers. Find the minimum value of $a_1 + a_2$.

Solution

The sequence is defined in such a way as to preclude us from exploring the first few terms. However, it seems unlikely that the particular choice of 1887 materially alters the behaviour of such a sequence. This suggests that it could be fruitful to consider the differences of terms at the start of the sequence. The definition gives

$$a_3(a_2 + 1) = a_1 + 1887, \quad a_4(a_3 + 1) = a_2 + 1887, \quad a_5(a_4 + 1) = a_3 + 1887. \qquad (1)$$

Subtracting the first pair of adjacent equations gives $a_4a_3 - a_3a_2 + a_4 - a_3 = a_2 - a_1$. This may be rearranged as

$$a_3 - a_1 = a_3(a_4 - a_2) + a_4 - a_2 = (a_3 + 1)(a_4 - a_2). \qquad (2)$$

Treating the next pair in a similar fashion, we have

$$a_4 - a_2 = a_4(a_5 - a_3) + a_5 - a_3 = (a_4 + 1)(a_5 - a_3). \qquad (3)$$

Note that (2) and (3) have started the separation of terms into those of odd index and those of even index. Consider the first few terms: how are they behaving? Suppose that $a_3 - a_1 \neq 0$. Then

by (2), $a_4 - a_2 \neq 0$. (Make quite sure you understand this statement: remember all the terms of the sequence are ***positive.***) It now follows from (3) that $a_5 - a_3 \neq 0$ and so on. (Of course you could formalise this into a full inductive argument if you wished.)

Continuing the process started in (2) and (3), we have

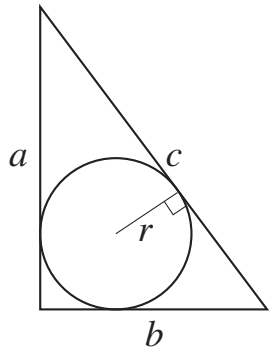
$$|a_{n+3} - a_{n+1}| = \frac{|a_{n+2} - a_n|}{|a_{n+2} + 1|}. \tag{4}$$

But $|a_{n+2} + 1| \geqslant 2$. (Here remember again how the terms are defined.) It follows that

$$|a_{n+3} - a_{n+1}| = \frac{|a_{n+2} - a_n|}{|a_{n+2} + 1|} < |a_{n+2} - a_n|. \tag{5}$$

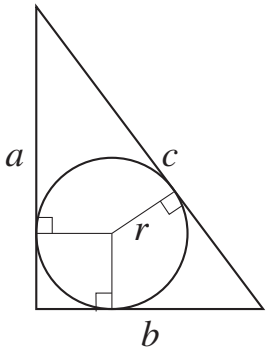
Applying (5) to the start of the sequence gives $|a_3 - a_1| > |a_4 - a_2| > |a_5 - a_3| > \cdots$. This is a ***contradiction.*** (If this does not seem obvious, then ask yourself where this sequence of differences will end up.) Returning to our original assumption, we see that all the terms with an odd index are equal and the same is true of the terms with an even index. The definition of the sequence then implies that $a_1 = a_3 = \dfrac{a_1 + 1887}{1 + a_2}$. Therefore $a_1 a_2 = 1887 = 3 \times 17 \times 37$. We can now answer the question posed: the minimum value of $a_1 + a_2$ is $51 + 37 = 88$.
Further question: What has 1887 to do with this problem sheet?

5. A right-angled triangle has integer side lengths a, b and c. Prove that the radius of the incircle (shown as r in the diagram below) is also an integer.

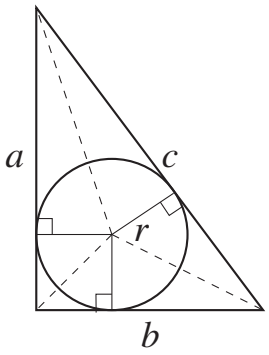


Solution

The theorem of Pythagoras will be useful here so let us write down $a^2 + b^2 = c^2$.
Dropping all possible perpendiculars gives the diagram below.



This diagram contains some obvious kites and adding some dashed lines, as below, reveals some useful triangles.



Now we see two ways to express the area of the original triangle:

$$\frac{1}{2}ab = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr \quad \text{which simplifies to} \quad ab = ar + br + cr.$$

By making c the subject and substituting it into the Pythagoras result, we arrive, after some manipulation, at the following quadratic equation:

$$r^2 - (a+b)r + \frac{ab}{2} = 0.$$

The solutions are $r = \dfrac{a+b \pm \sqrt{(a+b)^2 - 2ab}}{2}$. However, under the square root we have $a^2 + b^2 = c^2$ and so the solutions reduce to $r = \dfrac{a+b \pm c}{2}$. As r is the radius of the incircle, it must be smaller than each of the three sides. Therefore the positive root must be discarded to give $r = \dfrac{a+b-c}{2}$.

To finish the proof, it remains to show that $a + b - c$ is even. When we consider the Pythagorean triples that give the three sides of the original triangle, we can restrict ourselves to the so-called primitive ones. These are the triples where a, b and c share no factor other than 1. When they do share a factor greater than 1, then r shares that same factor. All primitive triples contain two odd integers and one even integer but this statement merits further discussion.

If a and b are both even, then $a^2 \equiv 0 \pmod 2$ and $b^2 \equiv 0 \pmod 2$. Hence $c^2 \equiv 0 \pmod 2$. But this means that a, b and c share a factor of 2, ***contradicting*** our requirement of a primitive triple.

For the next case we can, without loss of generality, take a to be odd and b even. Then $a^2 \equiv 1 \pmod 2$ and $b^2 \equiv 0 \pmod 2$. It follows that $c^2 = a^2 + b^2 \equiv 1 \pmod 2$. There is no contradiction here and we have an example in the most famous triple of all.

If both a and b are odd, then $a^2 \equiv 1 \pmod 2$ and $b^2 \equiv 1 \pmod 2$. Hence $c^2 = a^2 + b^2 \equiv 2 \pmod 2$. However, if you review the Pythagorean triples that you know, then you will realise that this does not describe what happens. You may recollect that, in similar situations, one strategy is to change the base. An inviting choice is to work modulo 4, because squares are congruent to 0 or 1 modulo 4. (If you are not sure about this, then run through the possibilities before proceeding.) Now we have that $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$. (Again, before proceeding, check that you believe this.) Hence $c^2 = a^2 + b^2 \equiv 2 \pmod 4$ and this furnishes our ***contradiction***.

As a result of this discussion, we know that

$$a + b - c \equiv 1 + 0 - 1 \equiv 0 \pmod 2.$$

It follows that $r = \dfrac{a+b-c}{2}$ is an integer, as required.

6. Prove that, for positive a, b, c and d,

$$\frac{a}{a+2b+3c+4d} + \frac{b}{b+2c+3d+4a} + \frac{c}{c+2d+3a+4b} + \frac{d}{d+2a+3b+4c} \geqslant \frac{2}{5}.$$

Solution

Initial impressions of this problem are that (a) there are a lot of variables in it and (b) most appear in the denominators of the algebraic fractions. Thus a possible start on understanding the problem is to (a) consider an analogue with fewer variables and (b) essentially invert the fractions. To this end, reduce the problem to

$$\frac{a}{a+2b} + \frac{b}{b+2a} \geqslant F, \tag{1}$$

where F is a suitable fraction. Also set $u = a + 2b$ and $v = b + 2a$. Rearranging gives $a = \dfrac{1}{3}(-u + 2v)$ and $b = \dfrac{1}{3}(2u - v)$ so

$$\frac{a}{a+2b} + \frac{b}{b+2a} = \frac{1}{3}\left(\frac{-u+2v}{u} + \frac{2u-v}{v}\right) = \frac{2}{3}\left(\frac{u}{v} + \frac{v}{u} - 1\right) \tag{2}$$

But, by the two-variable AM-GM inequality, $\dfrac{1}{2}\left(\dfrac{u}{v}+\dfrac{v}{u}\right) \geqslant \sqrt{\dfrac{u}{v} \times \dfrac{v}{u}} = 1$ or equivalently $\dfrac{u}{v}+\dfrac{v}{u} \geqslant 2$. Substitution in (2) gives

$$\frac{a}{a+2b} + \frac{b}{b+2a} \geqslant \frac{2}{3}\left(2 - 1\right) = \frac{2}{3}. \tag{3}$$

This preliminary work shows that the method is promising but, in the problem given, it will be harder to express the fractions involved in terms of the variables analogous to u and v.

Moving to the given problem, set the sum of fractions on the left-hand side of the desired inequality equal to S. Also set

$$t = a + 2b + 3c + 4d, \tag{4}$$
$$u = 4a + b + 2c + 3d, \tag{5}$$
$$v = 3a + 4b + c + 2d, \tag{6}$$
$$w = 2a + 3b + 4c + d. \tag{7}$$

This is effectively a system of four linear equations in four unknowns and there is a standard method for solving such systems. The method can be streamlined by the use of matrices and is important in deriving a method for solving such sets of equations in general. In this case the standard method would start by subtracting suitable multiples of (4) from the other equations in order to remove the terms in a from them. Unfortunately this leads to some undesirable fractions

so an alternative is to adapt the method to achieve the same end more simply. One such adaptation is to subtract (6) from (5), (7) from (6) and (4) from (7). This produces the system

$$\begin{aligned} t &= a + 2b + 3c + 4d, && (8)\\ u - v &= a - 3b + c + d, && (9)\\ v - w &= a + b - 3c + d, && (10)\\ w - t &= a + b + c - 3d. && (11) \end{aligned}$$

Concentrate on the first three equations and subtract the last from each of them. This leads to a set of three equations, each without a term in a. They are

$$\begin{aligned} 2t - w &= b + 2c + 7d, && (12)\\ u - v - w + t &= -4b + 4d, && (13)\\ v - 2w + t &= -4c + 4d. && (14) \end{aligned}$$

Now note that (13) and (14) respectively give b and c in terms of d and the variables t, u, v and w. Substituting for b and c in (12) gives

$$2t - w = d - \frac{u - v - w + t}{4} + 2d - \frac{2v - 4w + 2t}{4} + 7d. \qquad (15)$$

Tidying up (15) yields an expression for d in terms of t, u, v and w. This may be substituted into (13) and (14) to obtain similar expressions for b and c. The final step is to use one of the equations to get a expression for a. These calculations eventually produce $40a = -9t + 11u + v + w$, $40b = t - 9u + 11v + w$, $40c = t + u - 9v + 11w$ and $40d = 11t + u + v - 9w$. (Before using these results, it is prudent to check that they do satisfy equations (4) to (7).)

We now have the following expressions for the algebraic fractions in S:

$$\begin{aligned} \frac{a}{a+2b+3c+4d} &= \frac{1}{40}\left(-9 + 11\frac{u}{t} + \frac{v}{t} + \frac{w}{t}\right);\\ \frac{b}{b+2c+3d+4a} &= \frac{1}{40}\left(-9 + \frac{t}{u} + 11\frac{v}{u} + \frac{w}{u}\right);\\ \frac{c}{c+2d+3a+4b} &= \frac{1}{40}\left(-9 + \frac{t}{v} + \frac{u}{v} + 11\frac{w}{v}\right);\\ \frac{d}{d+2a+3b+4c} &= \frac{1}{40}\left(-9 + 11\frac{t}{w} + \frac{u}{w} + \frac{v}{w}\right). \end{aligned}$$

In the reduced version of the problem we used the two-variable AM-GM inequality; here we are likely to need the four-variable version. *(For the general AM-GM inequality, see Hanna Neumann, Sheet 4, Question 4.)* To obtain S we will have to add the four equations displayed above. Of the algebraic fractions on the right-hand side, deal first with those having coefficient 11. By the AM-GM inequality

$$\frac{1}{4}\left(\frac{u}{t} + \frac{v}{u} + \frac{w}{v} + \frac{t}{w}\right) \geqslant \sqrt[4]{\frac{u}{t} \times \frac{v}{u} \times \frac{w}{v} \times \frac{t}{w}} = 1, \quad \text{which implies} \quad 11\left(\frac{u}{t} + \frac{v}{u} + \frac{w}{v} + \frac{t}{w}\right) \geqslant 44.$$

The other groups of fractions give

$$\frac{v}{t} + \frac{w}{u} + \frac{t}{v} + \frac{u}{w} \geqslant 4 \quad \text{and} \quad \frac{w}{t} + \frac{t}{u} + \frac{u}{v} + \frac{v}{w} \geqslant 4.$$

Now, adding the equations in the last display and using the inequalities above, we have

$$S \geqslant \frac{1}{40}\Big(-36 + 44 + 4 + 4\Big) = \frac{16}{40} = \frac{2}{5},$$

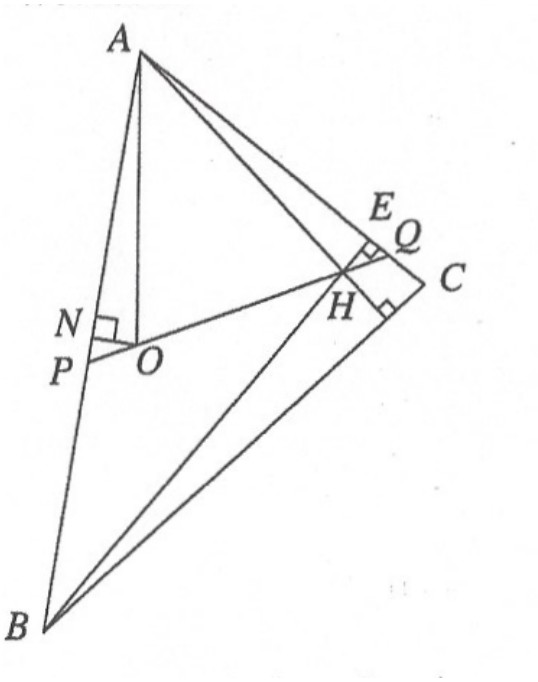
as required.

7. Let ABC be an acute-angled triangle with $AB > AC$ and $\angle BAC = 60°$. Denote the circumcentre by O and the orthocentre by H and let OH meet AB at P and AC at Q. Prove that $PO = HQ$.
Note: For the definition of orthocentre, see Srinivasa Ramanujan, Sheet 7, Question 3. The circumcentre is the centre of the circle which passes through the vertices A, B, C.

[British Mathematical Olympiad 2007 Round 2 Question 3]

SOLUTION

Triangle ABC is acute-angled and so both O and H lie inside it. Let the foot of the perpendicular from O to AB be N and let BH meet AC at E. In the usual manner we denote the angles of triangle ABC by $\angle A$, $\angle B$ and $\angle C$ and the sides by a, b and c. The configuration is shown below.



Now consider triangle EBA. We are given that $\angle EAB = 60°$ and so $EA = \frac{1}{2}AB = \frac{1}{2}c$. Since ON is the perpendicular from the centre of the circumcircle to the chord BA, it bisects BA and so $NA = \frac{1}{2}c$. Thus $NA = EA$.

Now $\angle BOA = 2\angle C$ (angle at the centre) and then, by symmetry, we have $\angle NOA = \angle C$. Also,

$$\angle AHE = 90° - \angle HAE = 90° - (90° - \angle BCA) = \angle C.$$

So since $NA = EA$ and, trivially, $\angle ANO = \angle AEH$, we find that triangle ONA is congruent to triangle HEA. Thus $HA = OA$, so triangle AOH is isosceles. We also have

$$\angle OAH = \angle CAB - \angle OAN - \angle HAE = 60° - (90° - \angle C) - (90° - \angle C) = 2\angle C - 120°.$$

Therefore, in the isosceles triangle,

$$\angle AOH = \angle AHO = 90^\circ - \frac{1}{2}\angle OAH = 90^\circ - (\angle C - 60^\circ) = 150^\circ - \angle C.$$

Consequently

$$\angle NOP = 180^\circ - \angle NOA - \angle AOH = 180^\circ - \angle C - (150^\circ - \angle C) = 30^\circ.$$

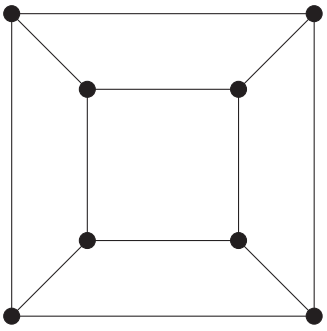
A similar argument shows that $\angle EHQ = 30^\circ$.

Finally consider triangles ONP and HEQ. We have just shown that $\angle NOP = \angle EHQ = 30^\circ$. Further, since triangle ONA is congruent to triangle HEA, $ON = HE$. Also $\angle ONP$ and $\angle HEQ$ are both right angles. Hence triangles ONP and HEQ are congruent. Hence $PO = HQ$, as desired,

8. A gardener wishes to place a garden soaker (which is a thick hose with small holes in it) in their plot, which consists of 8 white roses and 5 flower beds, shown below. The line segments represent the boundaries between the flower beds and also the outer boundary of the garden. Soakers can only be laid along these boundaries.

If soakers are laid such that each rose is passed exactly once, then the flower beds will be appropriately watered as well. The gardener possesses just one soaker. It is a closed curve which the gardener can stretch, compress, and bend, but not cut.

In how many different ways can the soaker be laid so that both the roses and the flower beds are watered appropriately?



Note: Even if you use a different terminology, be reassured that that the pipes are called "soakers" in Yorkshire, where the question setter lives.

SOLUTION

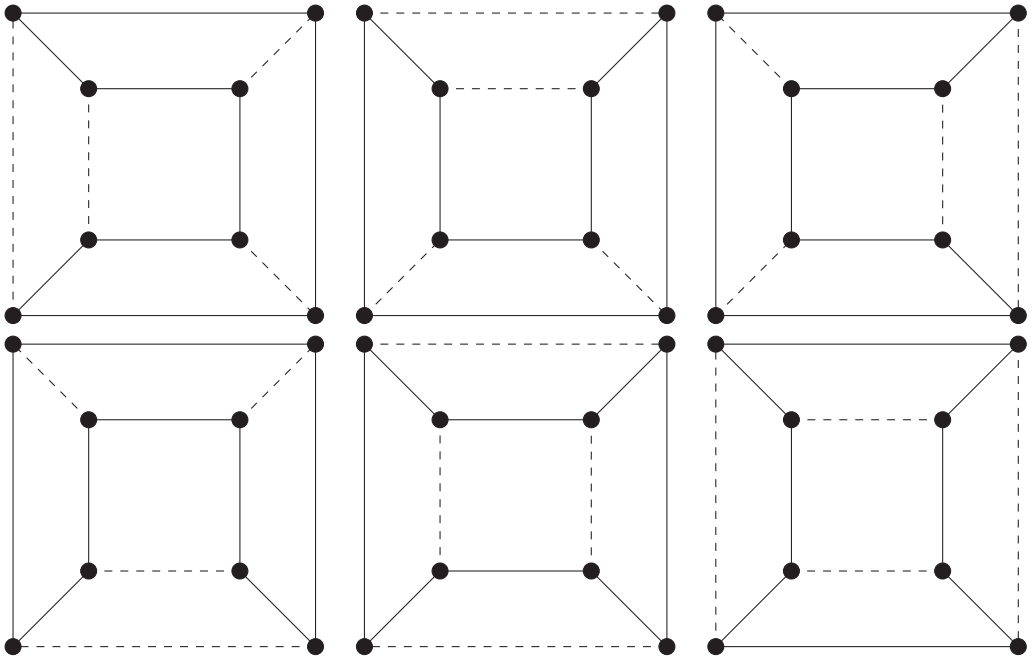
Since this is the last question of the last sheet, as a final flourish we present four different solutions here.

(a) ***Solution based on the idea of partition***.
If a closed cycle is laid surrounding certain number of flower beds, then the 5 flower beds will be partitioned into those who are inside the cycle and those who are outside. The possible partitions of 5 are (0,5), (1,4) and (2,3). We can analyse each situation according to the number of flower beds enclosed by the soaker.

- 0 flower bed: the trivial case, which is not possible.
- 1 flower bed: the soaker would pass by 4 roses only.

- 2 flower beds: the soaker would pass by 6 roses, as each flower bed is adjacent to 4 roses and any 2 neighbouring flower beds have 2 roses in common.
- 3 flower beds: this seems possible.
- 4 flower beds: if the centre flower bed is included, then we have 4 flower beds, each with 4 roses, and 5 shared boundaries. Therefore 10 roses are double counted. The soaker would pass by only 6 roses. If the centre flower bed were not included, then the design would require 2 separate soakers.
- 5 flower beds: the soaker would go through the outer boundary of the garden passing only 4 roses.

So it is worth proceeding only with situations where there are 3 flower beds inside the soaker and 2 outside. If the centre flower bed were to be outside, then one of its 4 neighbours would have to be outside as well. This gives us 4 different possibilities. If the centre flower bed were to be inside, then no 2 adjacent external flower beds could both be outside together. This arrangement would lead to 1 external rose not being visited by the soaker. This leaves 2 possibilities for the central flower bed to be inside. Overall this gives 6 different possible arrangements for laying the soaker. The diagrams below show that they can all be achieved.



(b) ***Solution based on coordinate geometry and symmetries***.
We assign coordinates to points in the given diagram. Say the four roses on the diagonal sloping upward and to the left have coordinates (0,0) (1,1), (3,3), (4,4).

Focus on (0,0). One of the edges leading to it must be omitted in a Hamiltonian cycle. Suppose (0,0)–(1,1) is not used. This means (1,1) must be connected to (3,1) and (1,3). There are now two choices for a rose to connect to (4,0): namely (3,1) or (4,4). For each case there is a unique way to complete the cycle.

If we had started with (0,4) instead, we would have arrived at the same two cycles, as the top-middle and the top-right solutions in the diagram above are reflections of each other

along the the upward sloping diagonal. Thus there are exactly two cycles where (0,0) is not joined to (1,1).

However, the graph can be visualised as a cube, so, by symmetry there must also be two cycles omitting (0,0)–(4,0), and two omitting (0,0)–(0,4). Thus, considering how (0,0) can be included, gives us a total of 6 distinct Hamiltonian cycles. Similarly, by considering how any vertex can be visited, we would obtain 6 Hamiltonian cycles. As we are searching for cycles passing through every single vertex exactly once, any Hamiltonian cycle based on a particular vertex appears in the set based on any other vertex. So overall the final answer is 6.

(c) ***Solution based on combinatorics***.
We can sort the edges into 3 sets: the outer square, the inner square and the four partial diagonals. Because of the nature of a cycle, if we go from one region to another, then we must be able to come back as well. This means we either use 2 of the partial diagonals or all 4 of them. If we use all 4 of them then we must use the edges of the inner and outer squares in pairwise combinations, horizontally for one and vertically for the other. This gives us 2 possibilities. If 2 of the partial diagonals are used, then they cannot be opposite to each other. We can see this because there are 2 paths of length 2 to connect them in either inner or the outer square. This arrangement allows the cycle to be outside or inside exactly once, so some vertices would not be visited. If we use 2 neighbouring partial diagonals and choose the path of length 3 that goes from one to the other on either square, then we can form a Hamiltonian cycle. There are 4 ways to choose such pairs, and there is only one path of length 3 to connect them in either square. So this gives us exactly 4 possibilities. Altogether there are 6 possibilities.

(d) ***Another solution based on combinatorics*** .
This graph is the planar graph of a cube. Imagine the planar graph of a polyhedron as follows. Suppose one of the faces has been removed and pretend that the remaining faces are made of rubber, so they can be pounded down to a make a two-dimensional flat surface. Thus the given graph has a face we cannot see.

In this particular graph each face has 4 vertices. This implies that each face must be contain 2 or 3 edges of a Hamiltonian cycle. Because if it contained 4 edges, then a cycle is formed that does not visit all the vertices. If it contained just one edge, then 2 vertices are not visited. As there are 8 vertices on this graph,we expect a Hamiltonian cycle to have length 8. But if we counted the total number of edges belonging to each face, we would obtain 16, as each edge belongs to 2 faces. There is only one way to form 16 by adding 6 numbers which are either 2's or 3's and that is $2 + 2 + 3 + 3 + 3 + 3$. Suppose we take any face and assign to it 2 edges of the Hamiltonian cycle. Then, in order for all 4 vertices of the face to belong to these 2 edges, the edges selected must be non-adjacent. From these 4 vertices the Hamiltonian cycle must continue on and at this stage 6 out of the 8 edges are assigned. In order to avoid creating 2 disjoint cycles, there is only one way to complete the cycle. This will result to all 4 faces adjacent to the initial face, to which we assigned 2 edges, having 3 edges of the Hamiltonian cycle. This would leave the last face, which is non-adjacent to the first face (in fact opposite to it) to have uniquely assigned two edges.

Therefore there are two distinct Hamiltonian cycles by assigning 2 edges of a Hamiltonian cycle to each face. There are 6 faces to a cube. However, the opposite faces produce the same cycle once 2 edges have been assigned. So the conclusion is that there are 6 distinct Hamiltonian cycles to a cube.

That completes the four solutions, showing once again how different parts of mathematics can be interconnected.